A MLM based on the $\Delta_3(L)$-statistic to correct for missed levels

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Motivation

- **Basic information about neutron resonance data**
- **Number of missed levels**
- **Use Random Matrix Theory**
- **Nearest Neighbor Distribution**

Can we use $\Delta_3(L)$

Data Sets

Nucleus	#levels	J
^{152}Sm	91	½
^{234}U	118	½
^{158}Gd	93	½
^{236}U	81	½
^{242}Pu	67	½
^{235}U	3160	3,4
^{58}Ni	63	½

http://t2.lanl.gov/cgi-bin/nuclides/endind

Acoustic resonances 6 sets of ≈ 250 levels

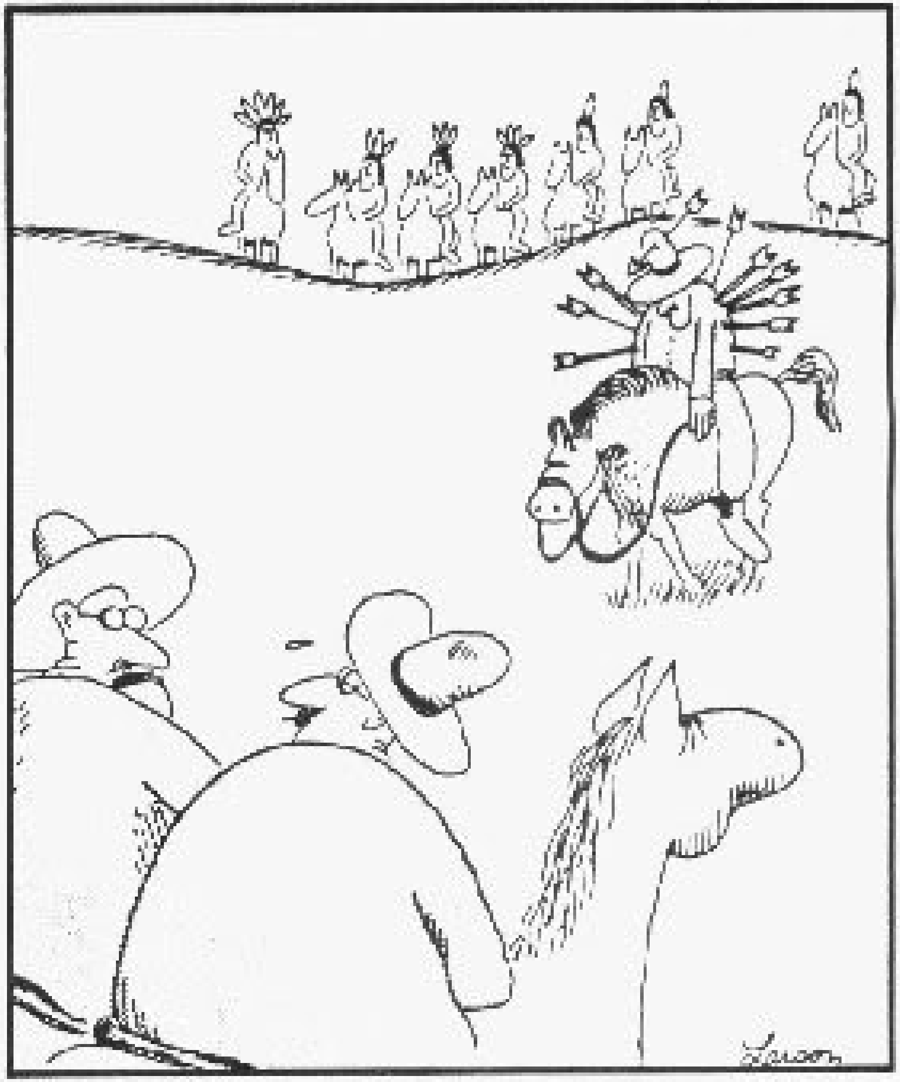
O. Antoniuk and R. Sprik, Journal of Sound and Vibration, 5489 (2010),

Raw Data

- Interpreted ENDF file for Pu-239e

- Resonance Parameters
- Number of isotopes represented: 1
- Isotope number: 1
- Isotope ZA: 94239.
- Isotope abundance: 1.0000
- Number of energy ranges: 4
- Energy range number: 1
- Lower energy limit: 1.000000-5
- Upper energy limit: 1.000000+3
- Reich-Moore Parameters
- Spin: 0.5
- Scattering length AP: 0.94100
- 4*pi*AP**2: 11.1273 barns
- Number of l states: 1
- Resonance Parameters for l= 0
- L-dependent scattering radius: 0.94100

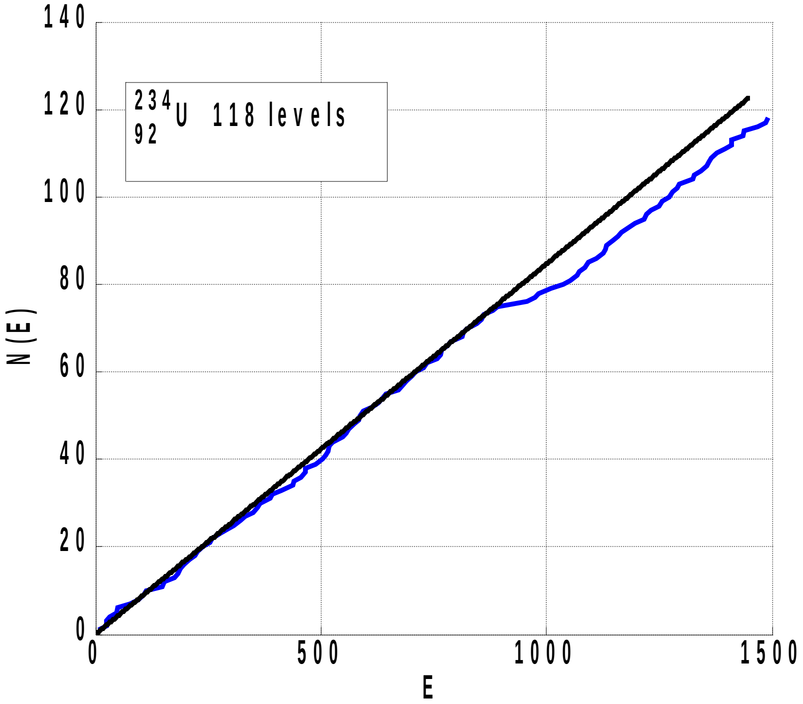
```
          eV          J          GN          GG          GFA          GFB
 ---------- ---------- ---------- ---------- ---------- ----------
-1.500200+2 1.000000+0 4.289000-1 4.572000-2 1.905000-1 0.000000+0
-1.546700+1 1.000000+0 1.355000-4 2.685000-2-2.553000-6 0.000000+0
-6.908700+0 0.000000+0 1.236000-2 2.600000-1-9.417000-1 2.962000-1
-2.194400-1 0.000000+0 3.047000-5 2.591000-3-1.614000-3-5.825000-1
 2.956243-1 1.000000+0 7.993000-5 3.930000-2 5.738000-2 0.000000+0
 7.815800+0 1.000000+0 7.920000-4 3.775000-2-4.475000-2 0.000000+0
 1.092800+1 1.000000+0 1.795000-3 3.612000-2-1.540000-1 0.000000+0
 1.189800+1 1.000000+0 9.751000-4 3.796000-2 2.071000-2 0.000000+0
 1.432900+1 1.000000+0 6.047000-4 2.921000-2 5.904000-2 0.000000+0
 1.467800+1 1.000000+0 1.910000-3 3.916000-2 3.045000-2 0.000000+0
 1.541700+1 0.000000+0 2.064000-3 4.200000-2-7.548000-6 7.550000-1
```

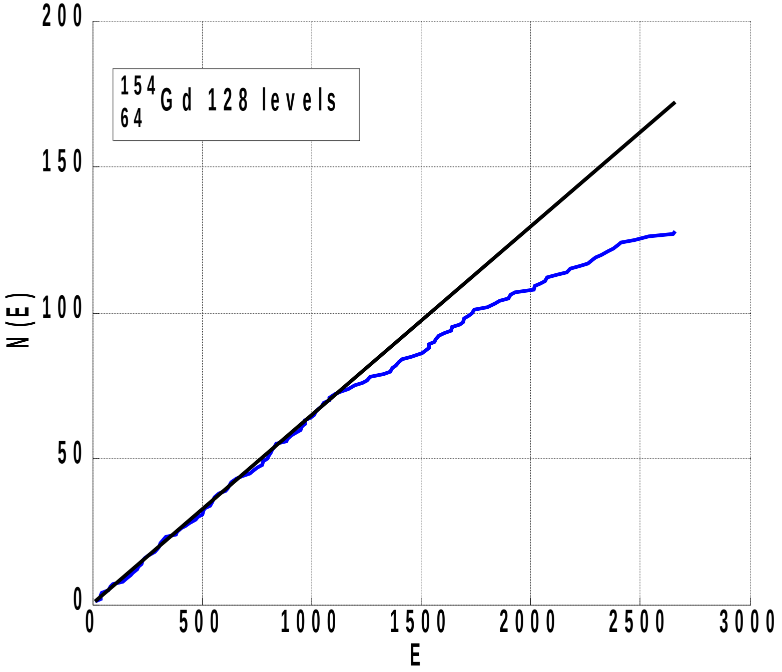


"Now stay calm. ... Let's hear what they said to Bill."

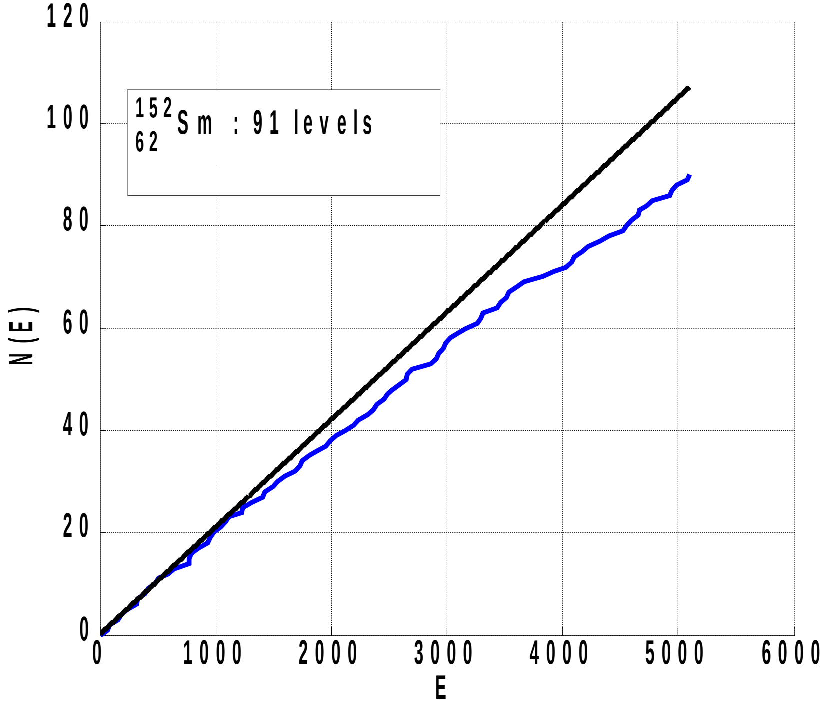
Don't miss the obvious

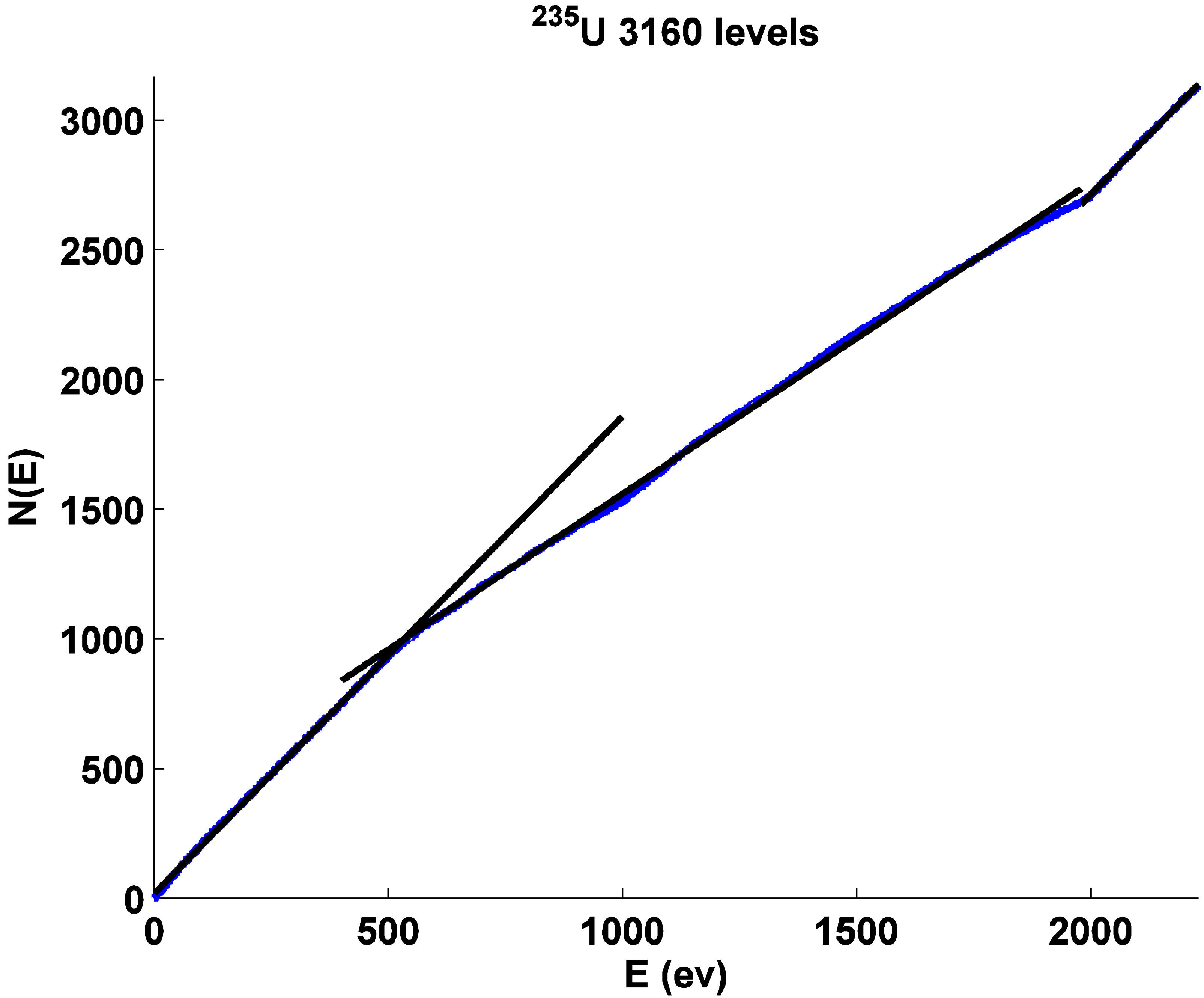
Cumulative Level Number: Raw Data





Cumulative Level Number: Raw Data





^{235}U 3160 levels
N(E)
3000
2500
2000
1500
1000
500
0
0
500
1000
1500
2000
E (ev)

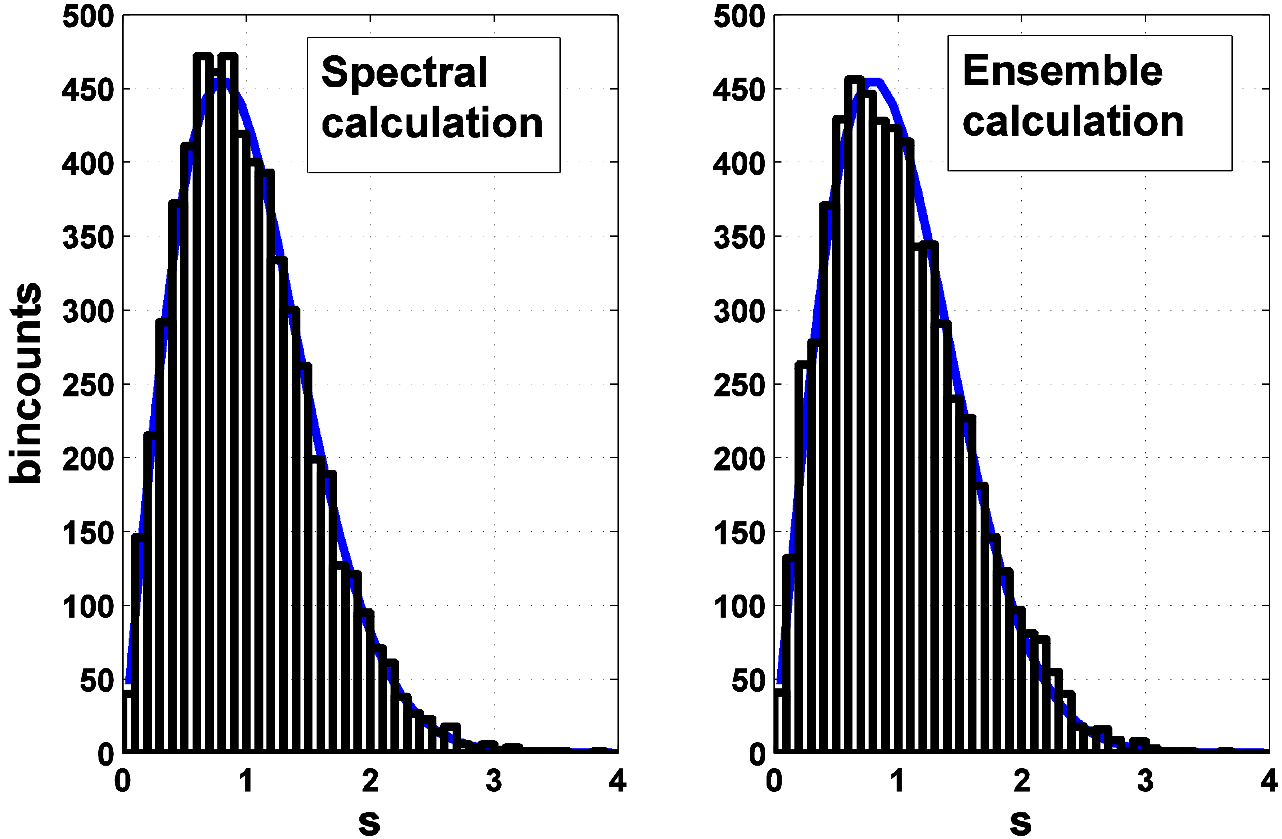
Level spacing

$$P(s) = \frac{\pi}{2} s e^{-\pi s^2/4}$$

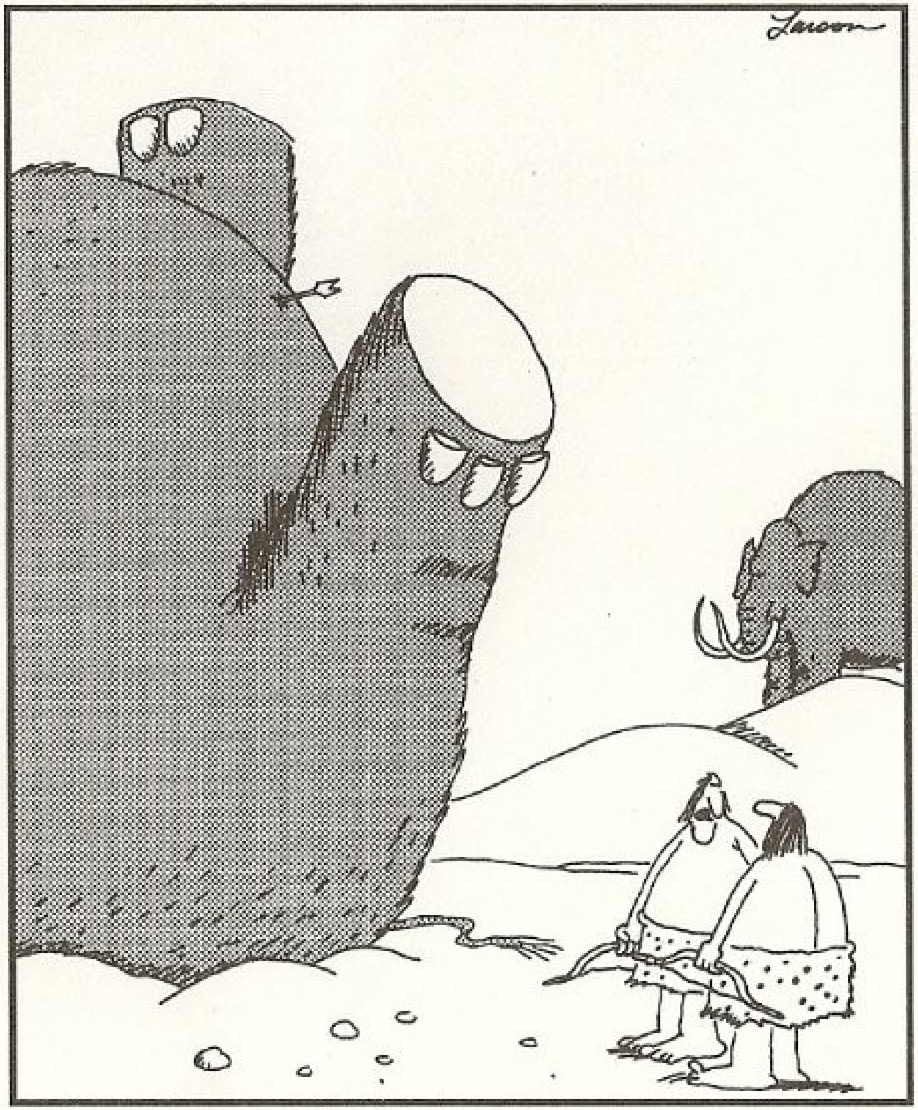
Statistic must be ergodic for RMT to apply

1 spectrum, N=6000
All spaces

6000 spectra (N=600)
Center space (E_{299}-E_{300})



Not so Ergodic



"We should write that spot down."

Maximum Likelihood Method based on p(s)

- **Level spacing distribution P(s) gives x, the fraction of missed levels. P(k;s) is k^{th} nearest level spacing**
- **Agvaanluvsan et al.Nucl. Instr. Meth. Phys. Res. A 498(2003)459-469**

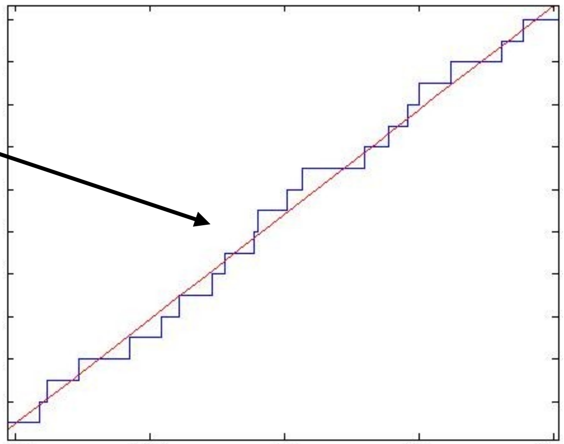
$$P(s) = \sum_{k=0}^{\infty} (1-x)x^k P(k;s)$$

$$L = \prod_i P(s_i)$$

Find the "x" that minimizes ln L

$\Delta_3(L)$ Statistic

- $\Delta_3(L)$ is the deviation of N(E) from a straight line
- Spectral ***rigidity***
- One spectrum, vary i



$$\Delta_3(L) = \left\langle \min_{A,B} \frac{1}{L} \int_{E_i}^{E_i+L} \left(N(E') - AE' - B\right)^2 dE' \right\rangle$$

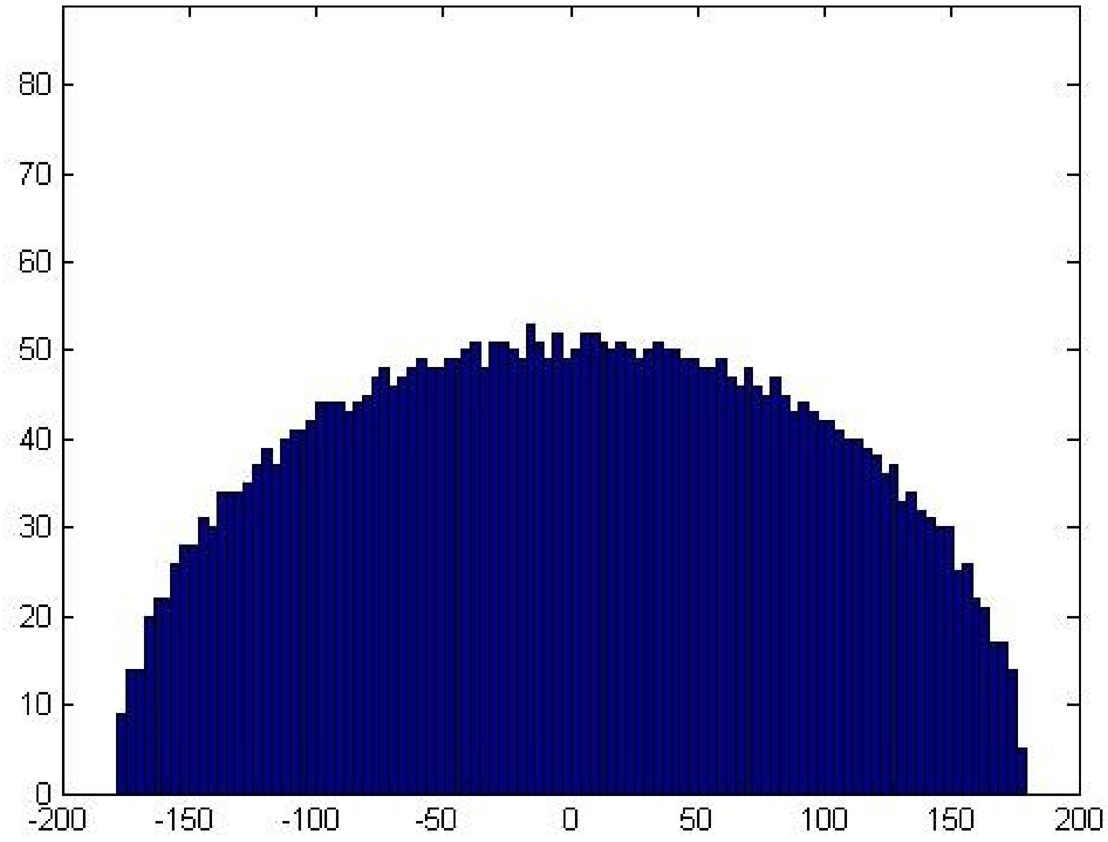
$$= \left\langle \Delta_3^i(L) \right\rangle_{spec}$$

The ‘<>’ brackets denote an average over the position, i, of window of length L, within the spectrum

A and B minimizes the area difference between the line and the unfolded spectrum

Unfolding removes secular variations in level density

Graph of energy density ρ(E) for GOE



GOE result for $\Delta_3(L)$

Dyson derived

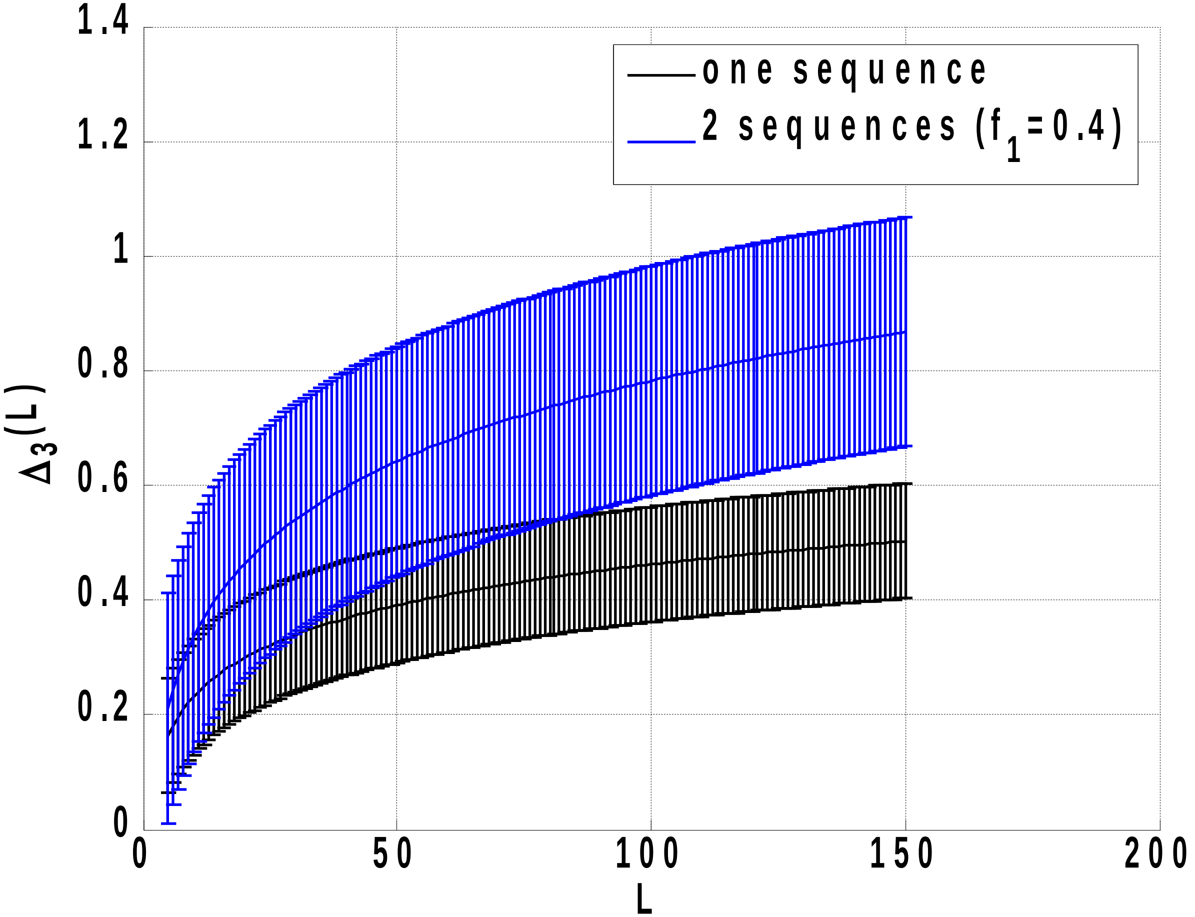
$$\Delta_3(L) = \frac{1}{\pi^2}\left(\log(2\pi L) + \gamma - \frac{5}{4} - \frac{\pi^2}{8}\right) = \frac{1}{\pi^2}\left(\log L - 0.0678\right)$$

With a standard deviation $\sigma_\Delta = 0.11$

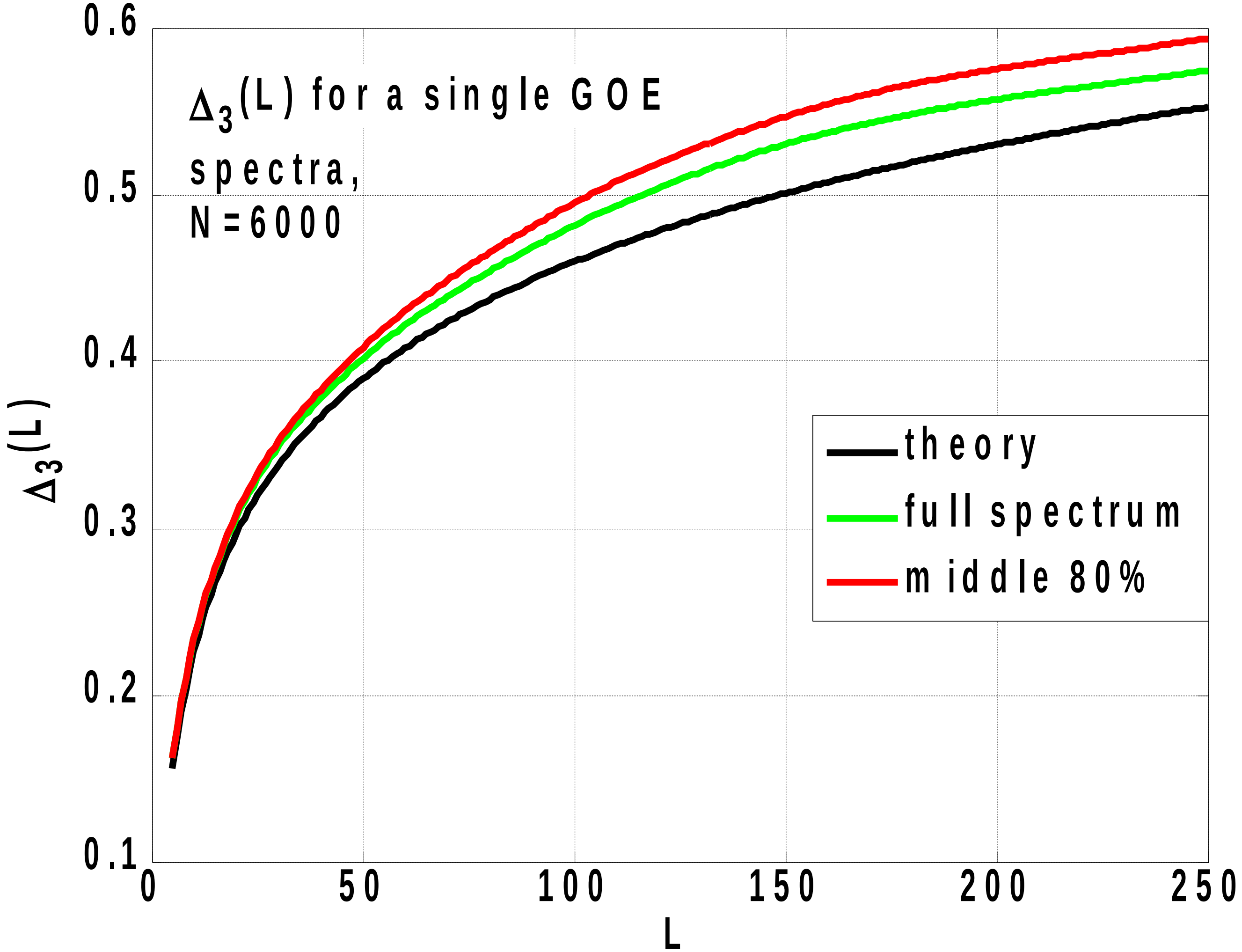
For m spectra superimposed in proportions $f_1:f_2:..f_m$

$$\Delta_3(L) = \sum_{i=1}^{m} \Delta_3(f_i L)$$ **and** $$\sigma_\Delta = 0.11\ \mathbf{m}$$

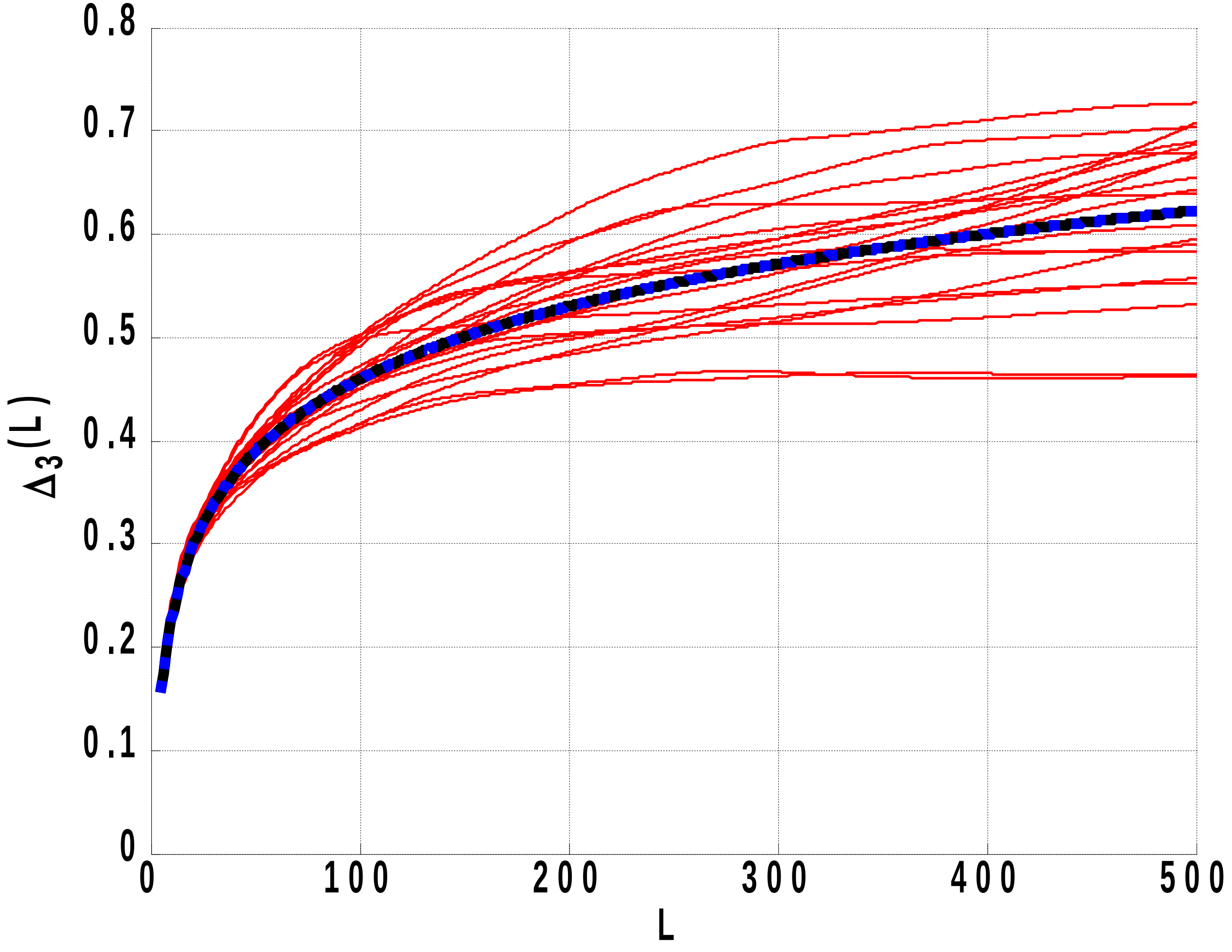
Error too big to be useful



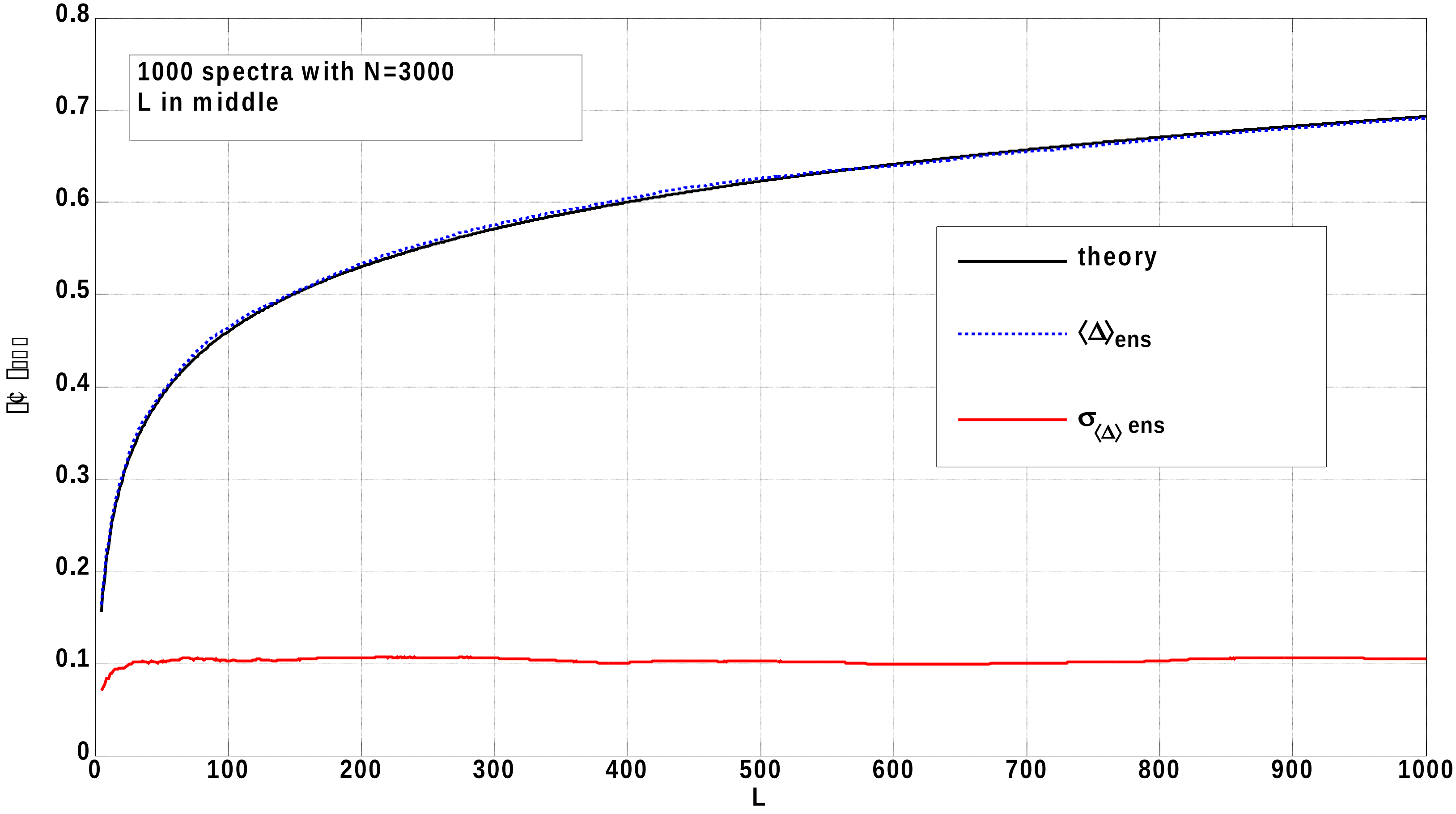
Calculation of $\Delta_3(L)$



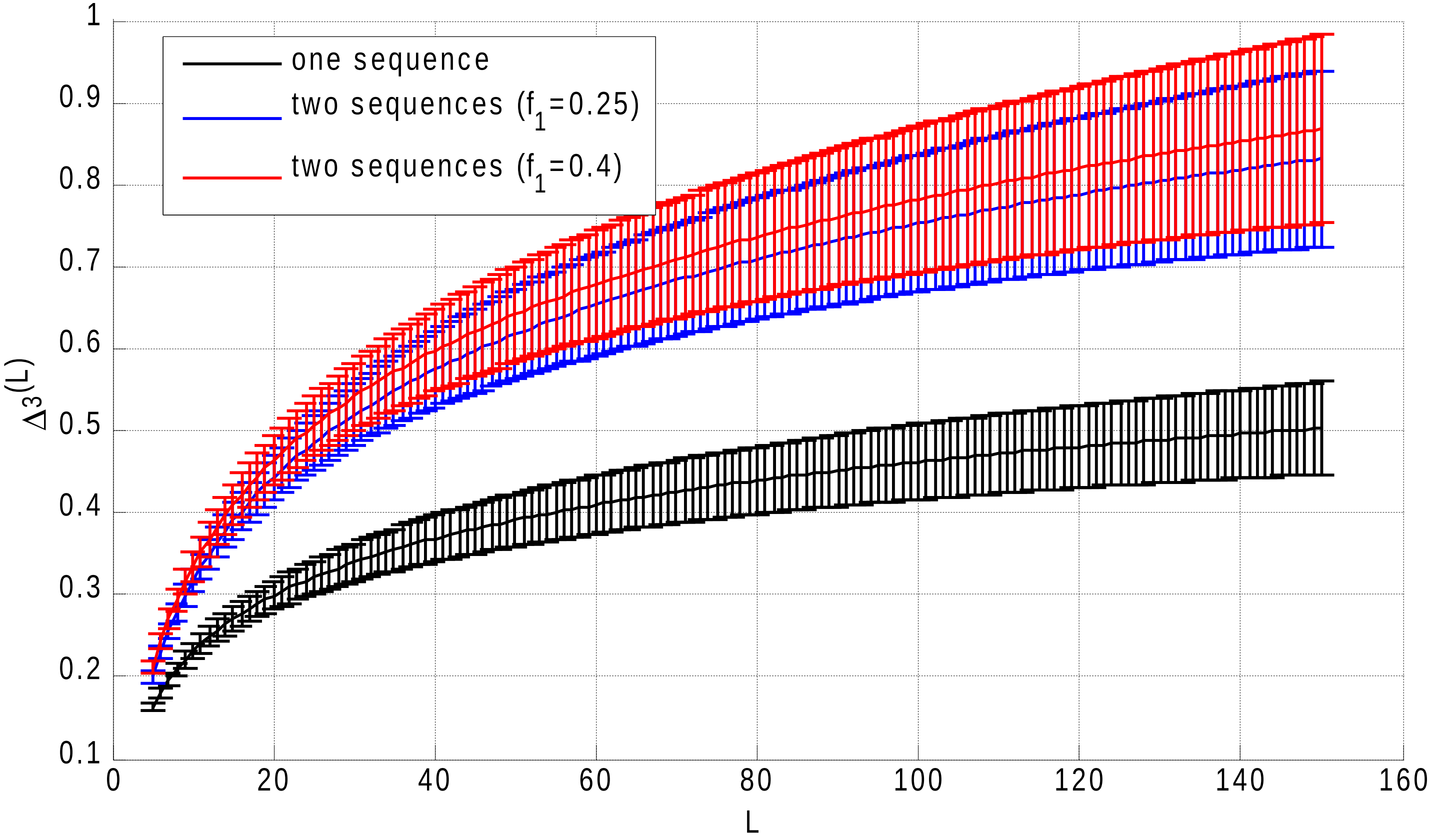
$\Delta_3(L)$ for many N=5000 spectra



Ensemble average, σ_{Δ}=0.11

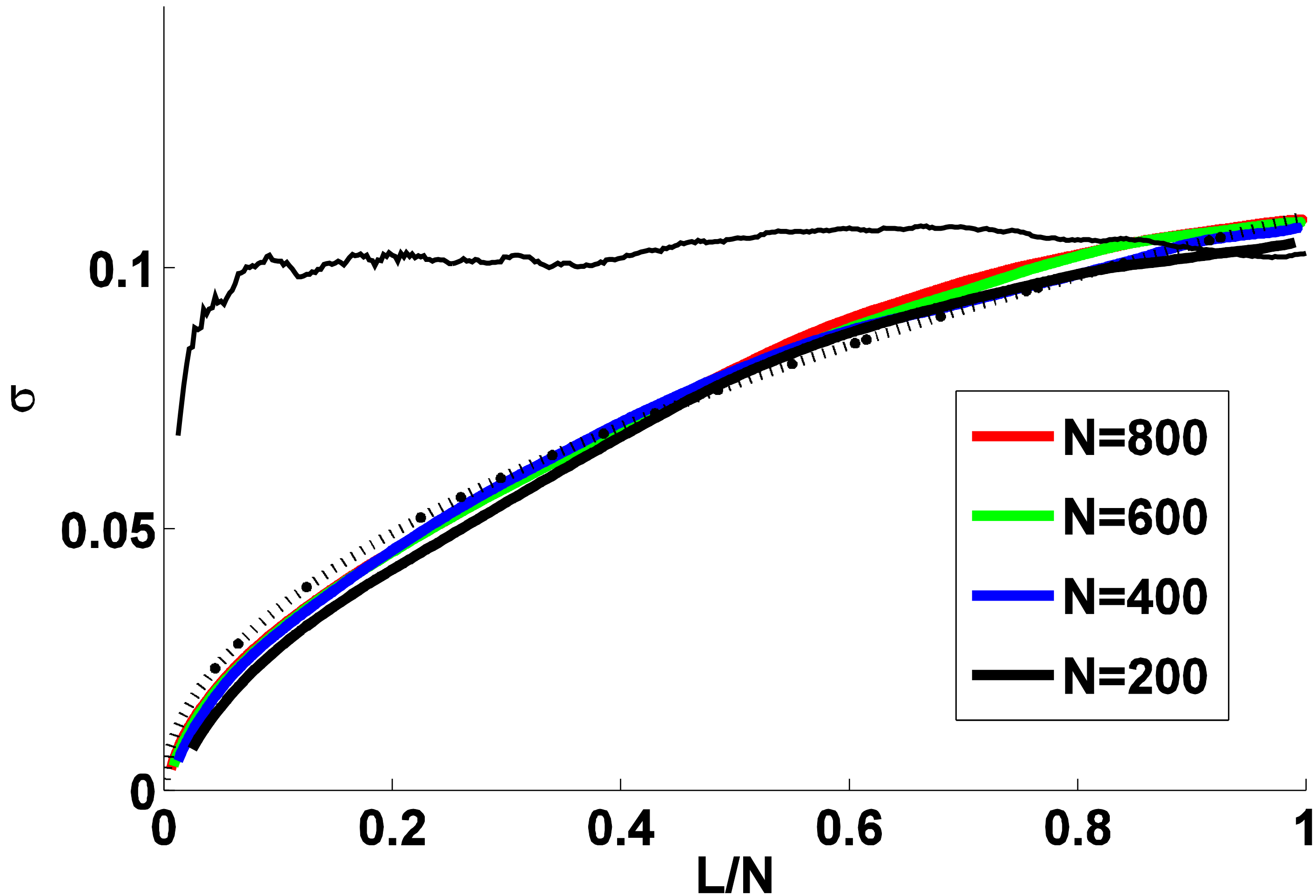


Empirical curves, with error bars



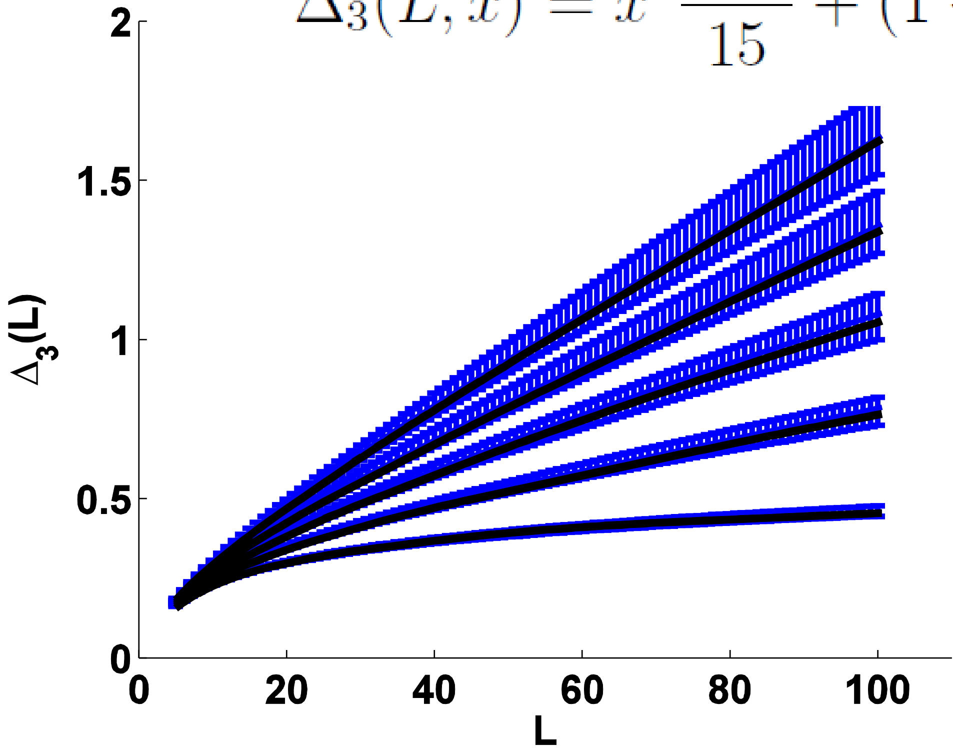
500 matrices
With N=500

error bars



Δ_3(L) for incomplete spectra

$$\Delta_3(L,x) = x^2 \frac{L/x}{15} + (1-x)^2 \Delta_3^{\mathrm{GOE}}(L/(1-x))$$



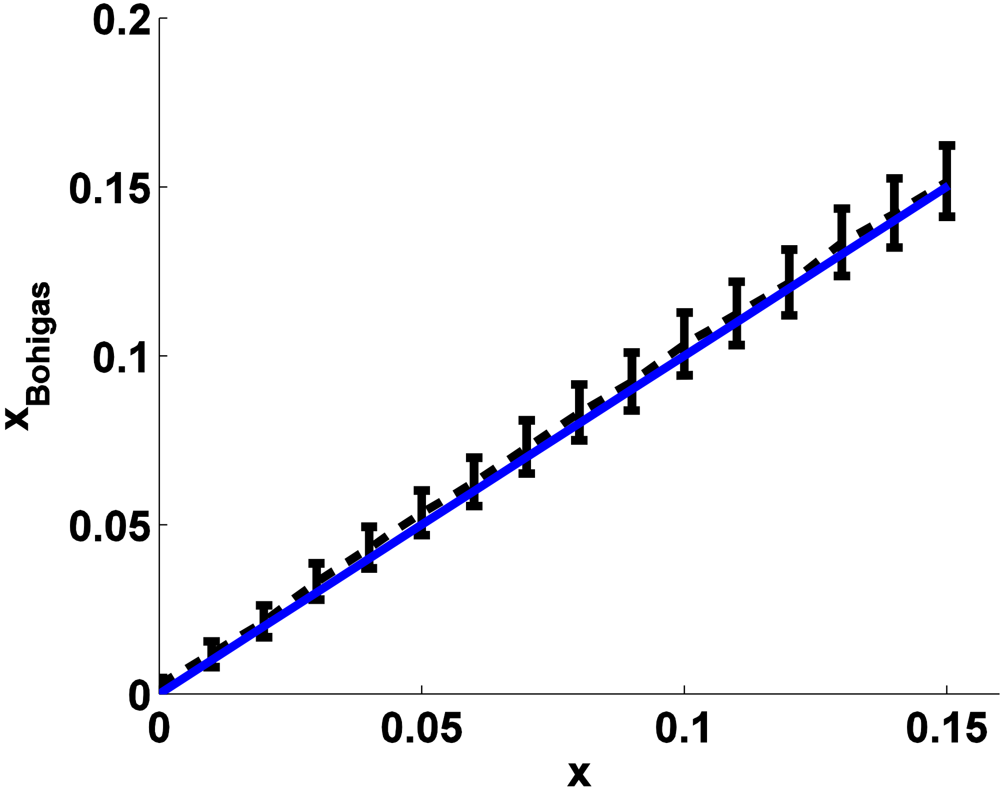
**Results for x=0%
5%,10%,15%,20%**

**Blue is GOE, 1200 spectra
N=10000**

O. Bohigas and M. P. Pato, Physics Letters B 595, 171 (2004),

Δ_3(L) for incomplete spectra

$$\Delta_3(L,x) = x^2\frac{L/x}{15} + (1-x)^2\Delta_3^{\mathrm{GOE}}(L/(1-x))$$



Agrees with p(s) analysis

Δ_3(L) Statistic again

$$\Delta_3(L) = \left\langle \min_{A,B} \frac{1}{L} \int_{E_i}^{E_i+L} \left(N(E') - AE' - B \right)^2 dE' \right\rangle$$

$$= \left\langle \Delta^i_3(L) \right\rangle_{ensemble}$$

$$i = \frac{N - L}{2}$$

This time the ‘<>’ brackets denote an average over many spectra while the position of the window, i, is fixed in the middle of the spectrum

New Approach

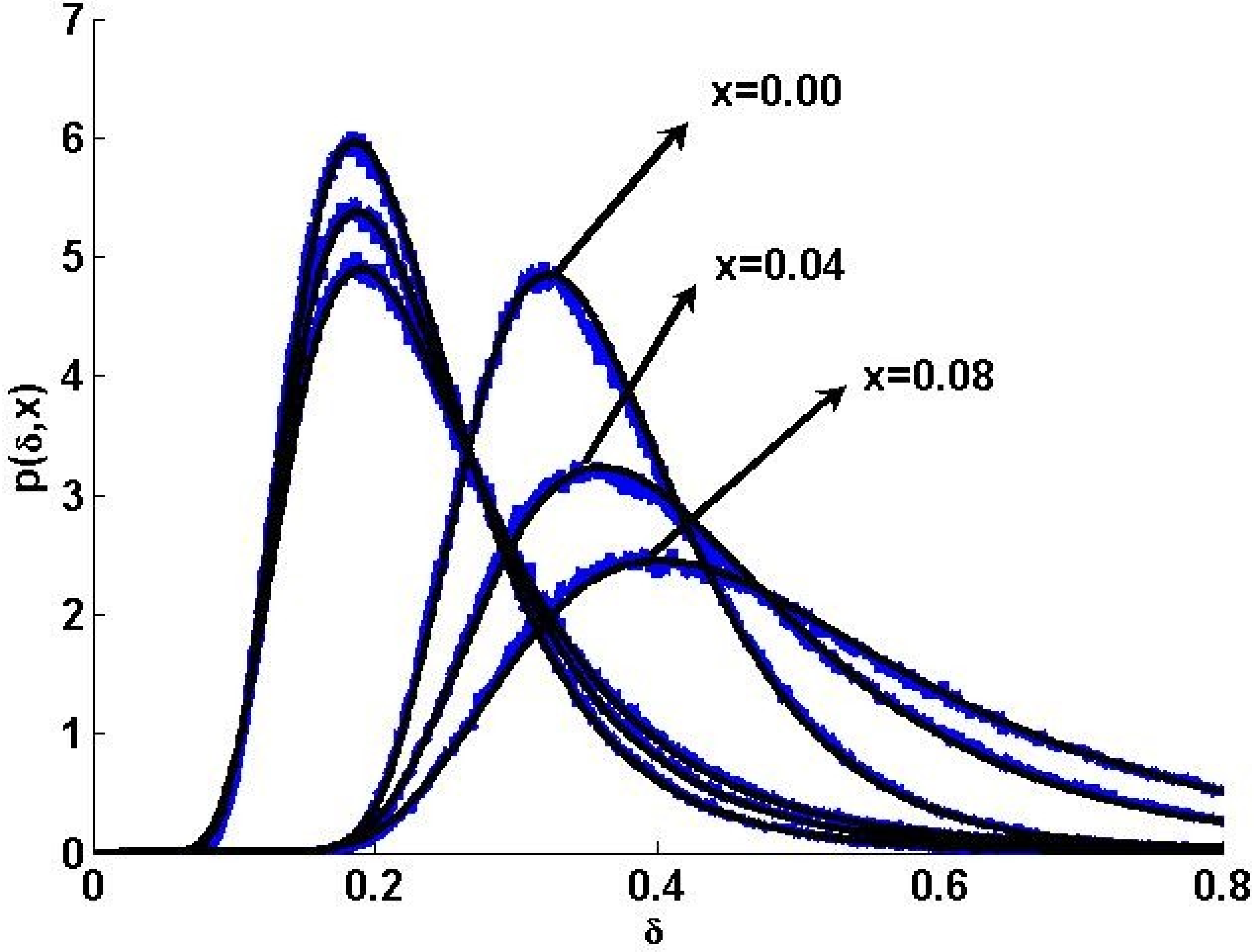
$$\Delta_3(L) = \left\langle \min_{A,B} \frac{1}{L} \int_{E_i}^{E_i+L} \left(N(E') - AE' - B \right)^2 dE' \right\rangle$$

$$= \left\langle \Delta^i_3(L) \right\rangle$$

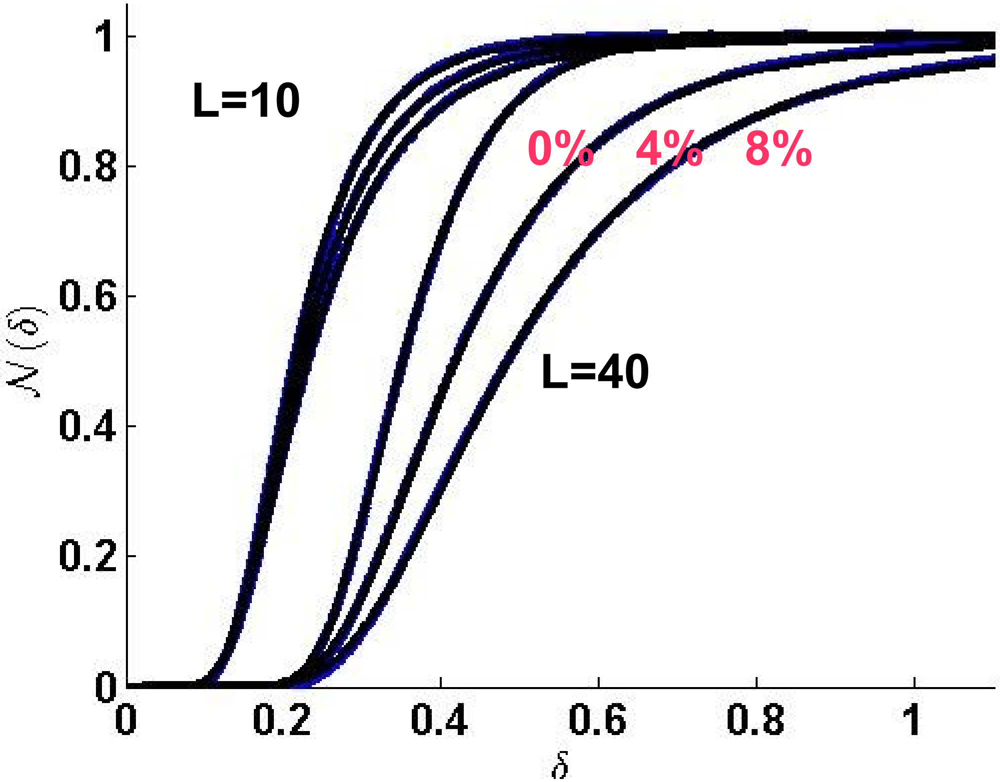
$$x = \Delta^i_3(L)$$

Look at the distribution of x within a spectrum

P(δ)



CDF of P(δ)



Parametrization
3 numbers do it

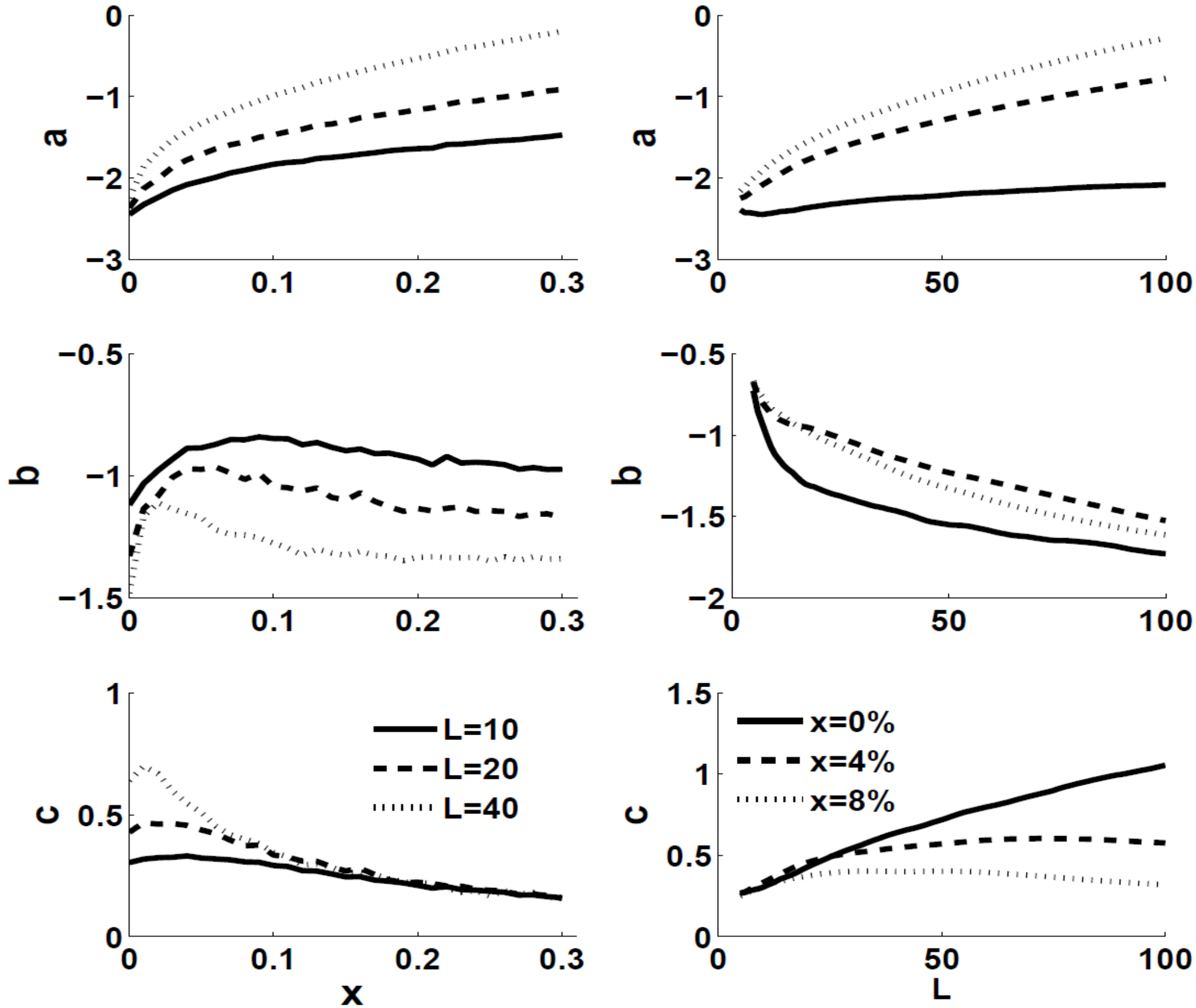
$$\mathcal{N}(\delta) = \frac{1}{2}(1 - \text{Erf}[a + b\log\delta + c(\log\delta)^2])$$

$$p(\delta) = -\frac{1}{\sqrt{\pi}}\exp\left[-(a + b\,\log\delta + c\log\delta^2)^2\right]\left(\frac{b}{\delta} + \frac{2\,c\log\delta}{\delta}\right)$$

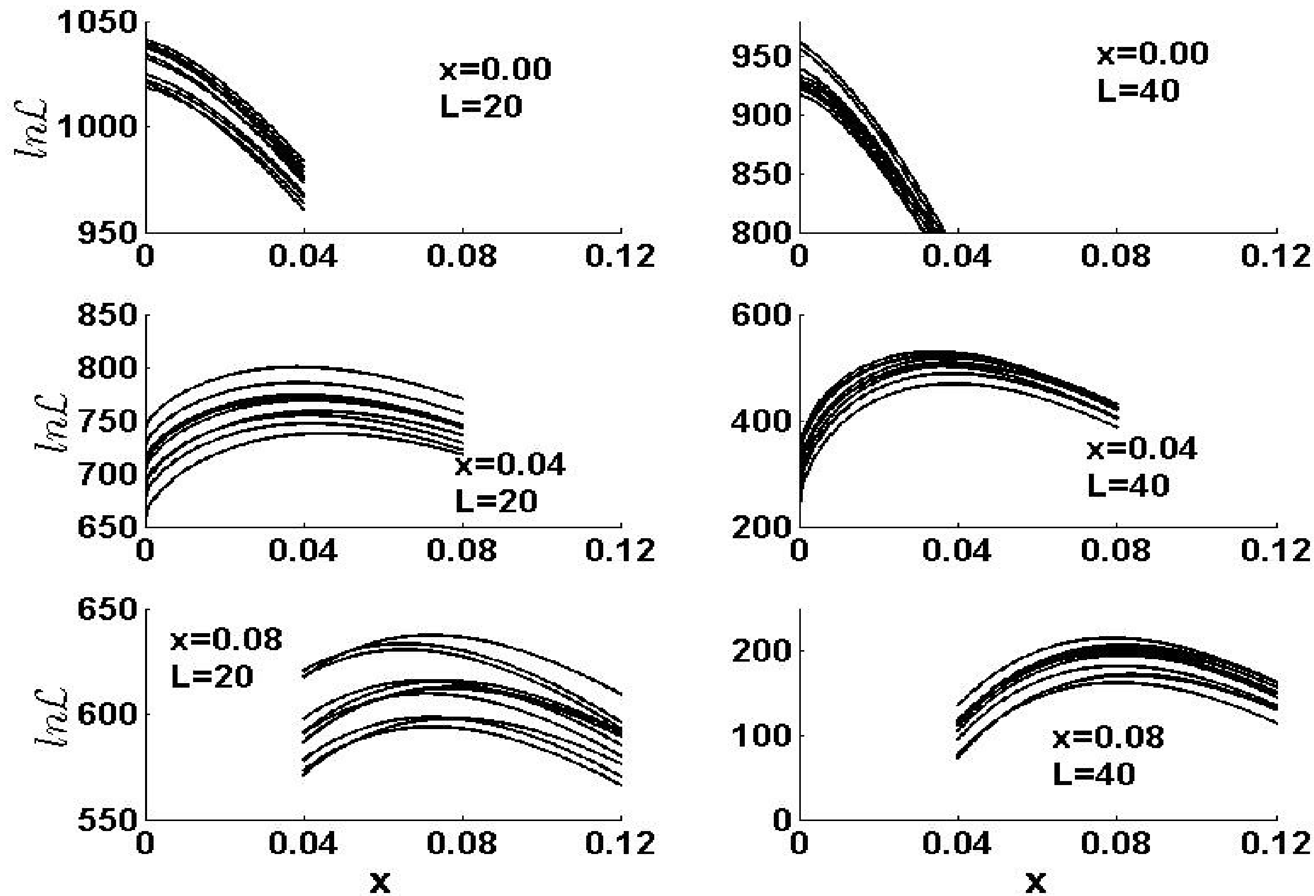
$$a_L(x) = a_0 + a_{\frac{1}{2}}\sqrt{x} + a_1 x + a_2 x^2$$

$$p(\delta, x) = -\frac{1}{\sqrt{\pi}}\exp\left[-(a_L(x) + b_L(x)\,\log\delta + c_L(x)\log\delta^2)^2\right]\left(\frac{b_L(x)}{\delta} + \frac{2\,c_L(x)\log\delta}{\delta}\right)$$

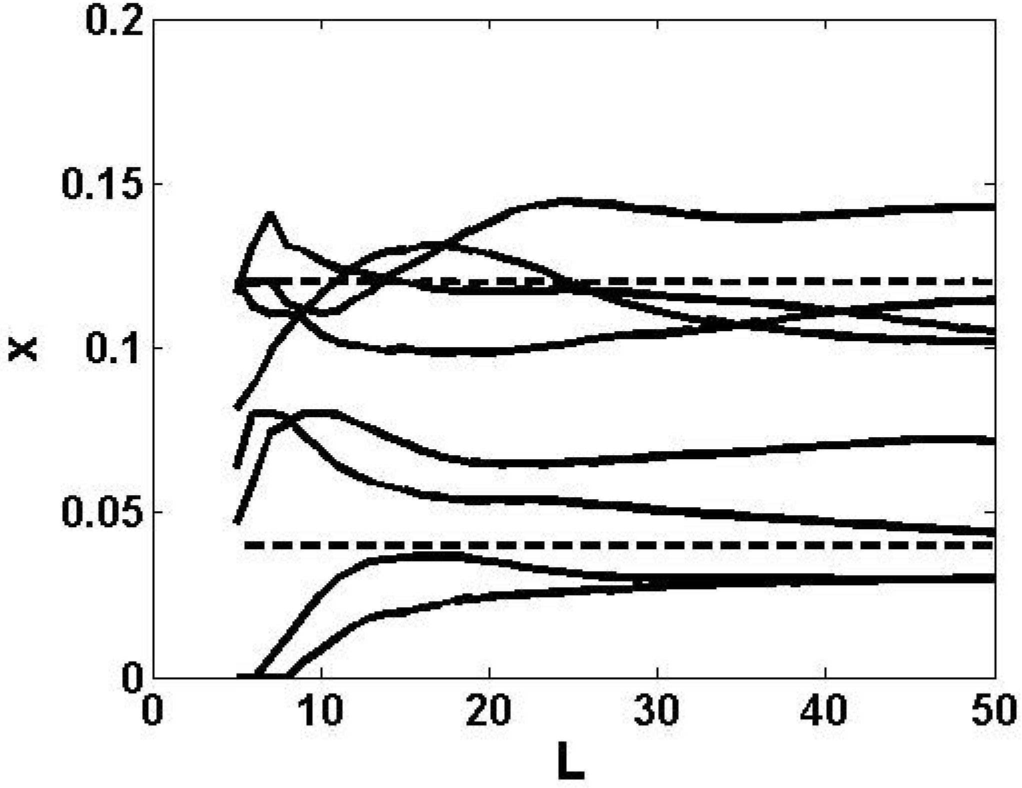
Parameters



Maximum Likelihood

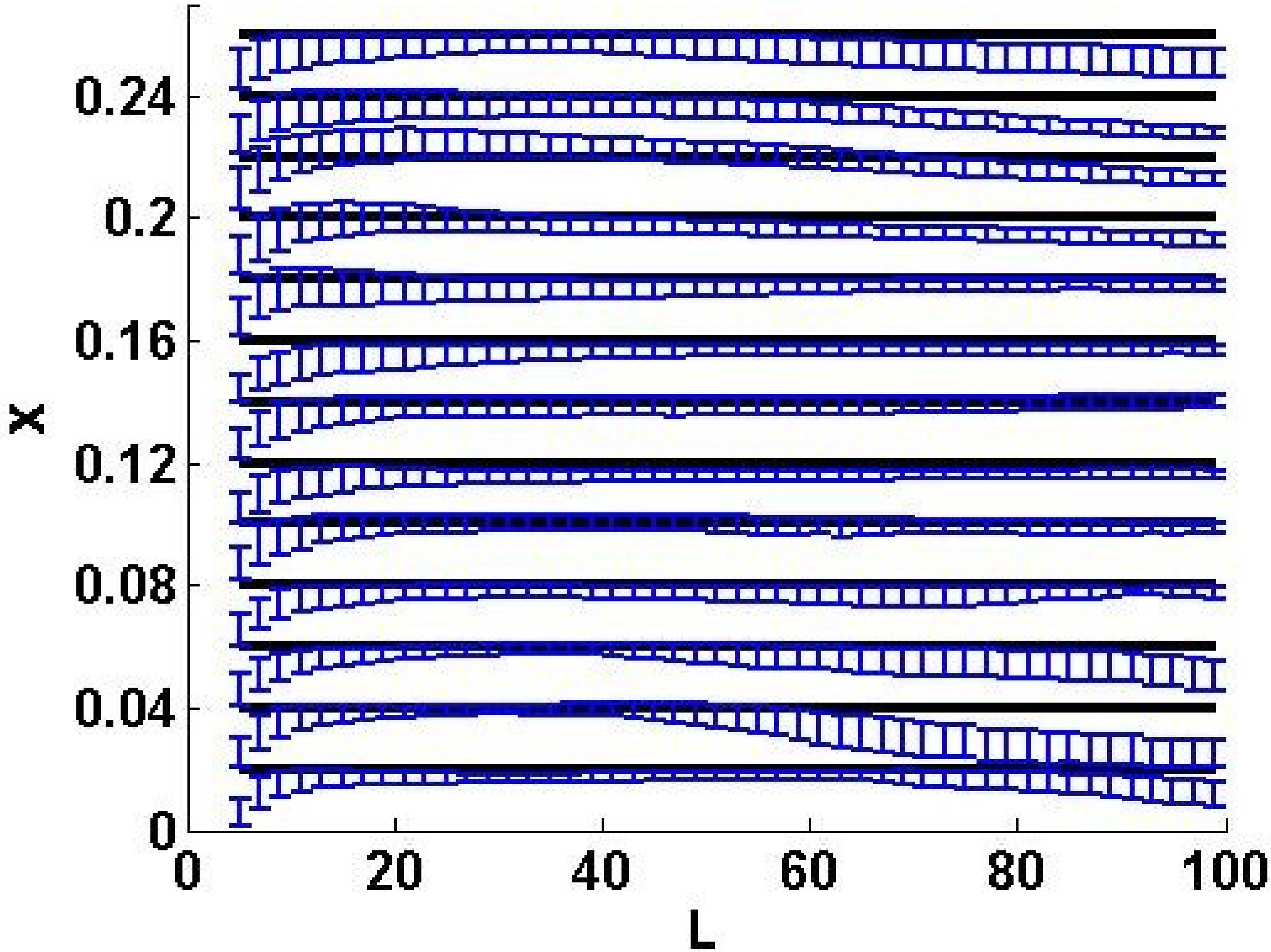


x vs L from MLM

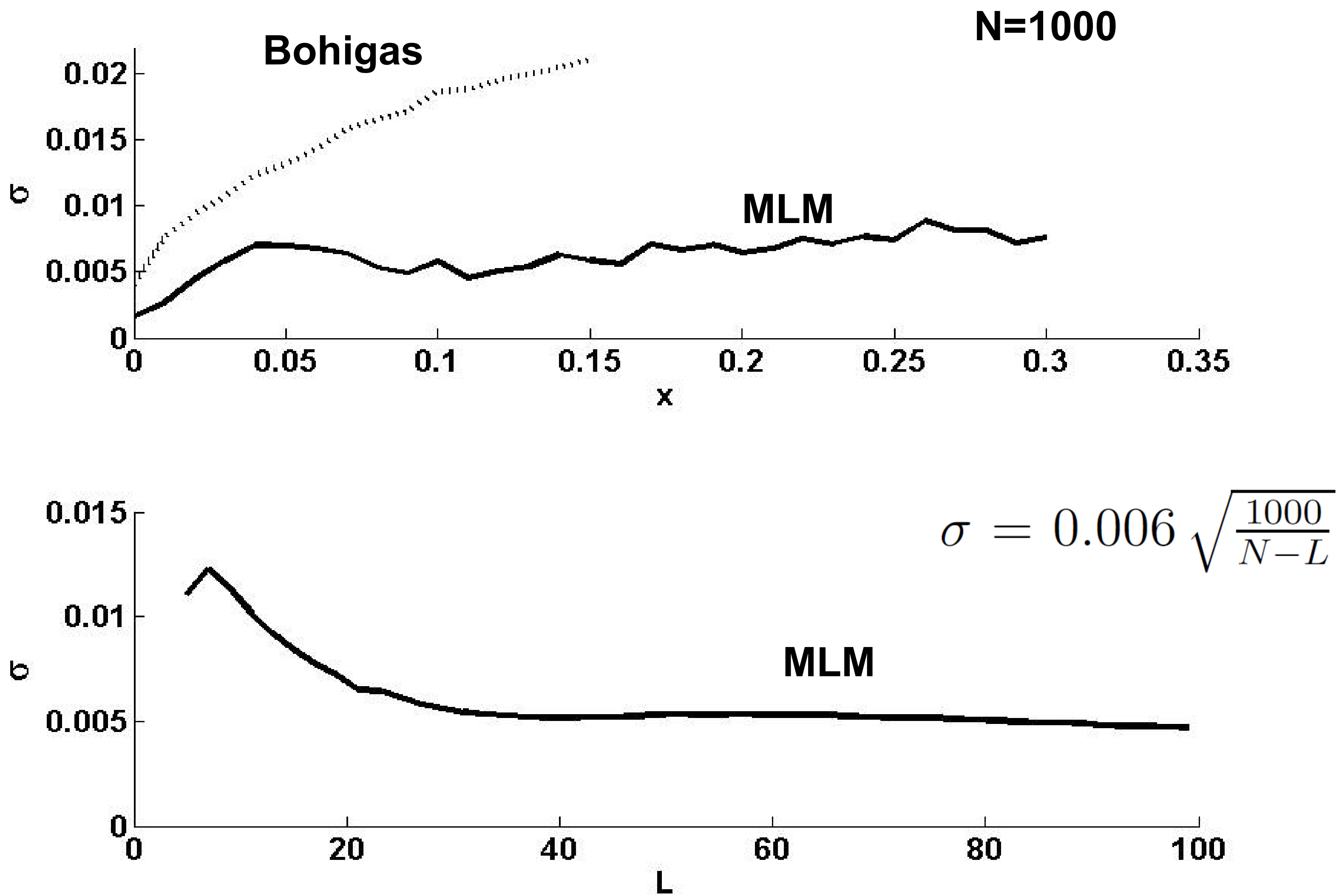


Results for some randomly chosen spectra with x=4% and x=12%

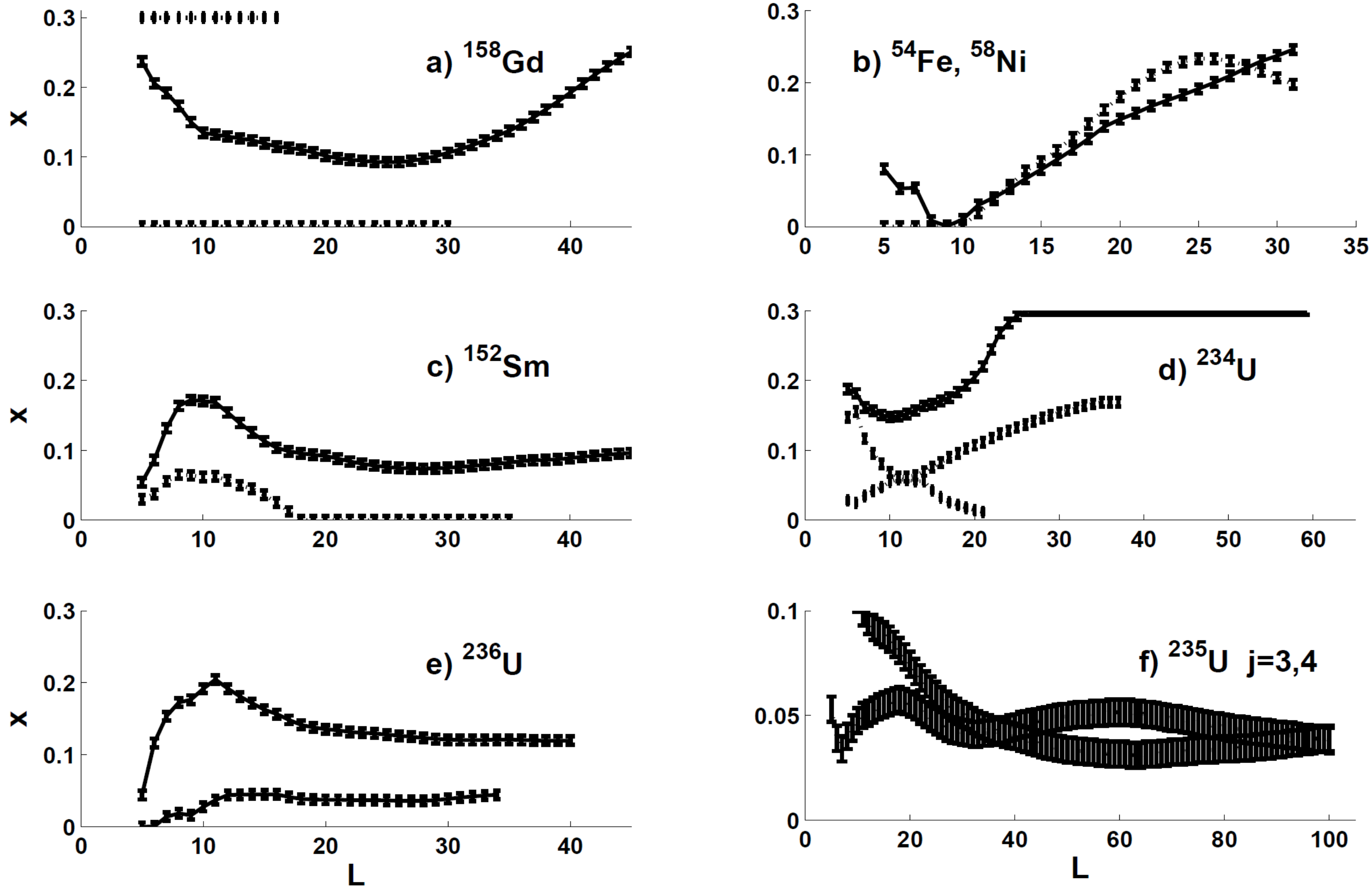
x vs L Ensemble results



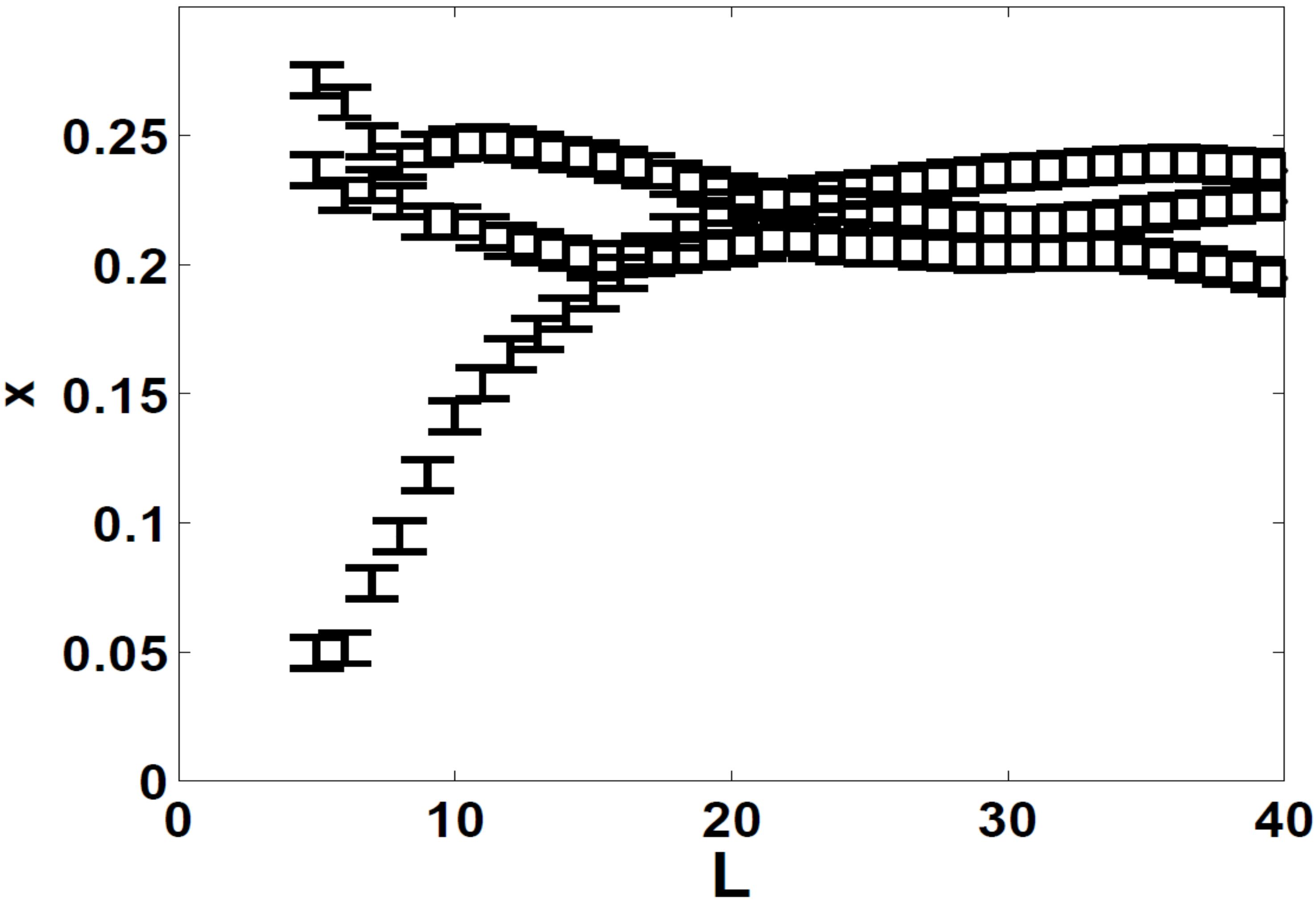
Average error bars



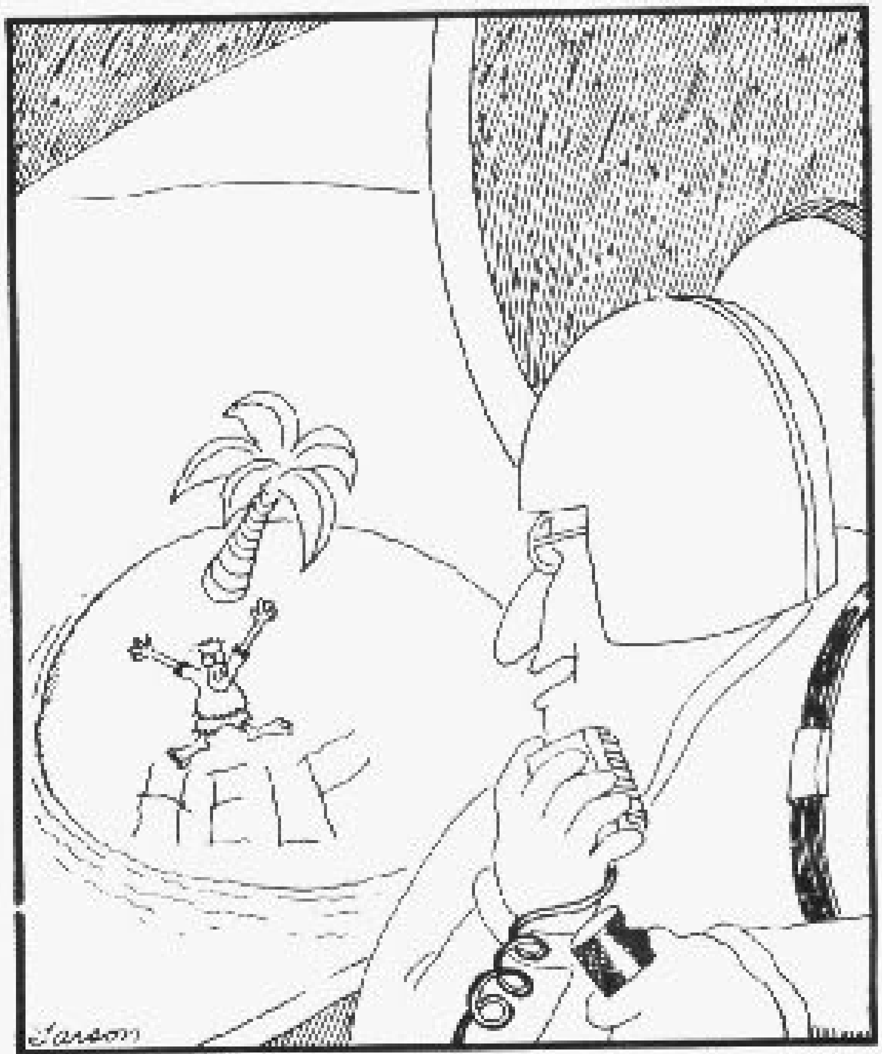
Isotopes



Acoustic Data



The mean values for the interval 20< L< 40 are 0.20, 0.22 and 0.23 respectively



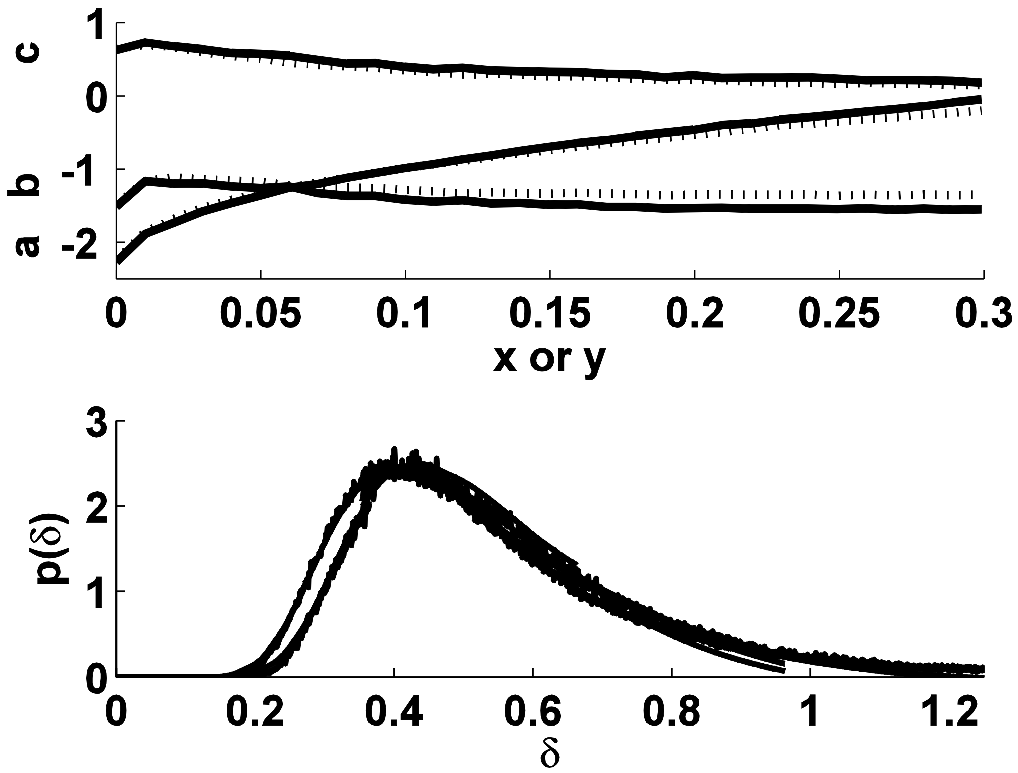
“Wait! Wait! Cancel that, I guess it says ‘helf.’”

Interpretation Can be tricky

TABLE I: The results for x, the percent of missing levels in the data.

Isotope	NND	$\Delta_3(L)$(Bohigeas)	$p(\delta)$	N (# levels)	subset
^{58}N	0%	18%	Inconclusive	63	All
^{152}Sm	3%	0%	0%	91	$1 \rightarrow 70$
^{152}Sm	3%	10%	8% ± 2%	91	All
^{158}Gd	11%	13%	12% ± 2%	93	All
^{158}Gd	0%	0%	0%	93	$1 \rightarrow 60$
^{158}Gd	12%	42%	>30%	93	$61 \rightarrow 93$
^{234}U	9%	40%	Inconclusive	118	All
^{234}U	6%	13%	Inconclusive	118	$1 \rightarrow 75$
^{234}U	7%	4%	Inconclusive	118	$76 \rightarrow 118$
^{236}U	5%	20%	12% ± 3%	81	All
^{236}U	0%	5%	4% ± 3%	81	$1 \rightarrow 69$
^{235}U $j = 3$	3%	9%	5% ± 1%	1436	$1 \rightarrow 381$
^{235}U $j = 4$	2%	4%	5% ± 1%	1732	$1 \rightarrow 569$

Intruder levels

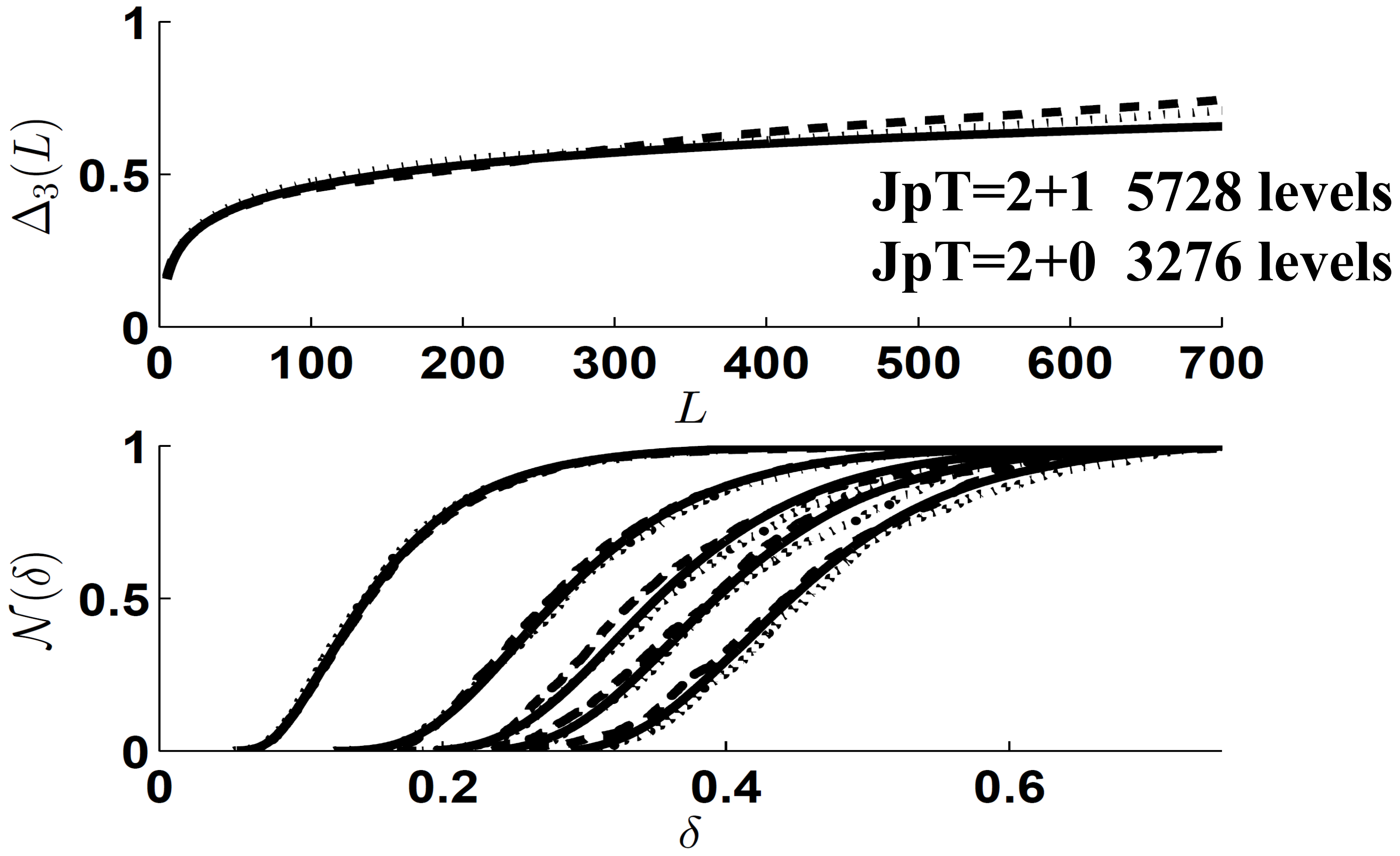


X=8% missed

Y=8% intruders

L=40

Shell Model:
12 particles-sd shell



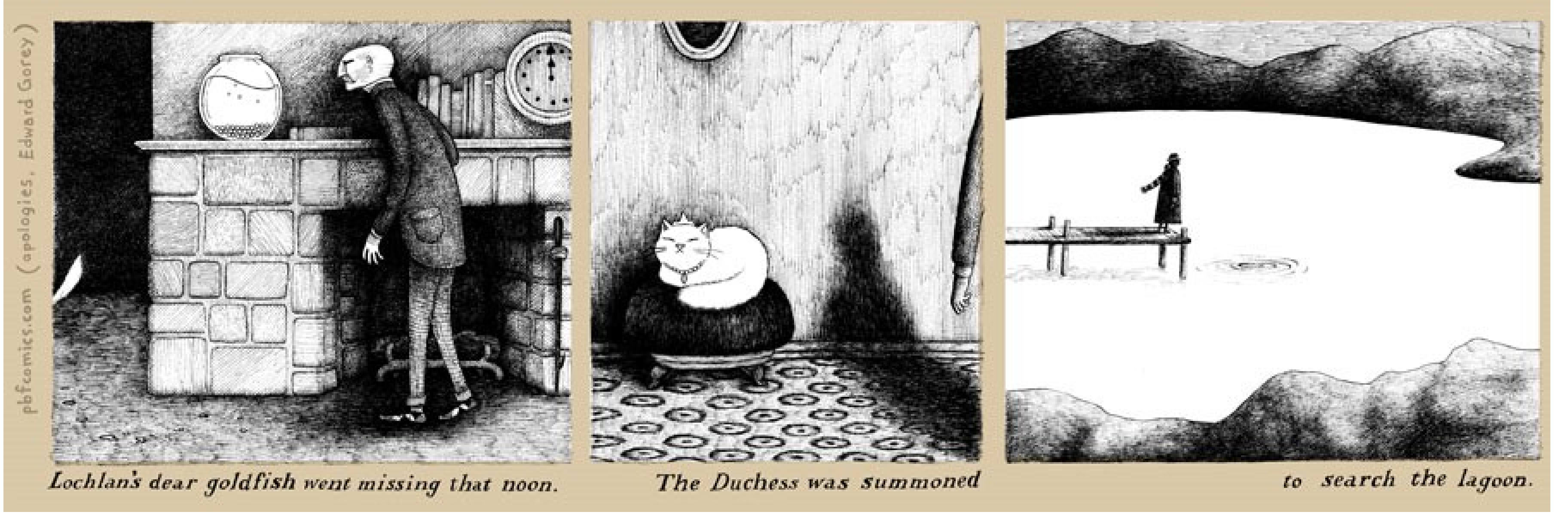
Conclusions

$\Delta_3(L)$ has “internal” distribution which gives a maximum likelihood method.

Results consistent with Bohigas expression

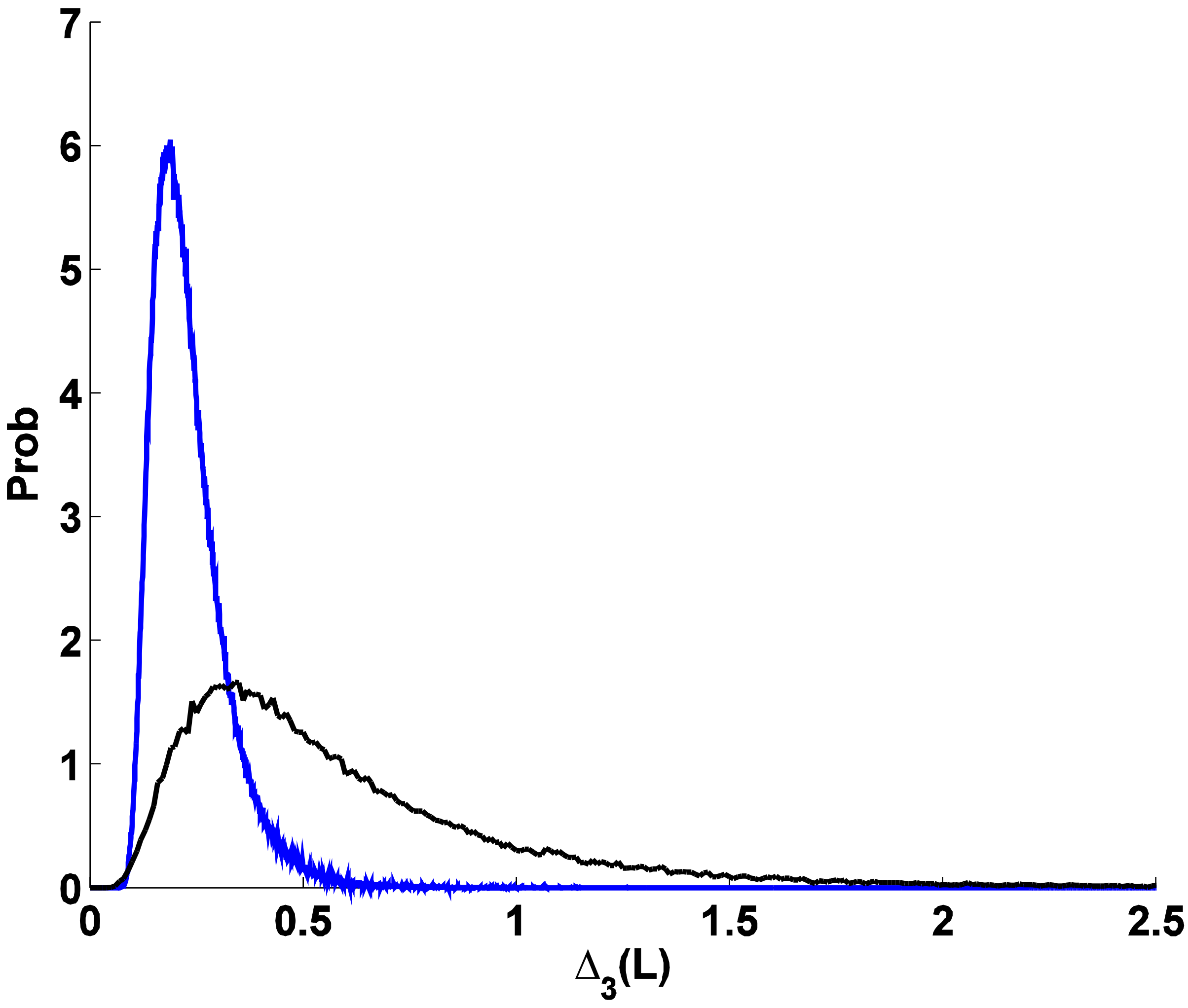
Intruders look like missed levels

Motivation cont'd



"Say . . . what's a mountain goat doing
way up here in a cloud bank?"

L=10; Poisson and GOE



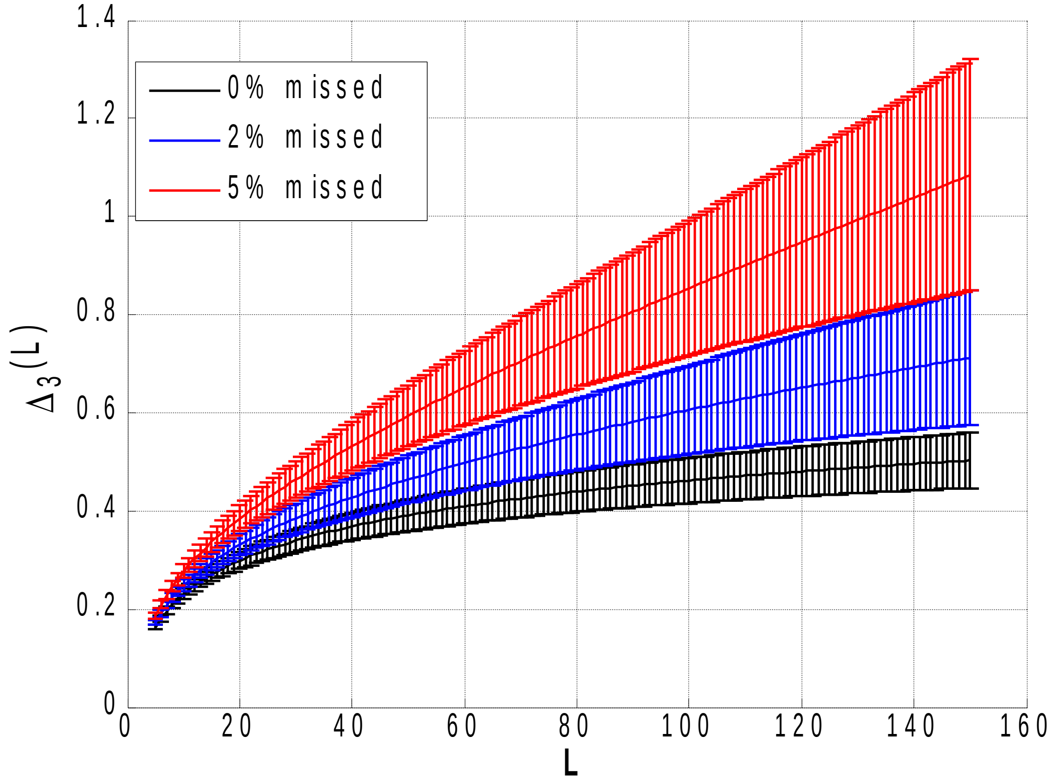
Analysis

- **Look at $\Delta_3(L)$ for real data and compare with GOE**

$$\chi^2(x) = \sum_{L\min}^{L\max} \frac{\left[\Delta_3(L) - \Delta_3(L;x)\right]^2}{\sigma(N,L;x)^2}$$

D. Mulhall, Z. Huard, and V. Zelevinsky, Physical Review C, 064611(2007)
2-level systems analyzed in D. Mulhall, Phys. Rev. C, 034612 (2009).

Empirical curves, with error bars N=500 levels



Maximum Likelihood Method

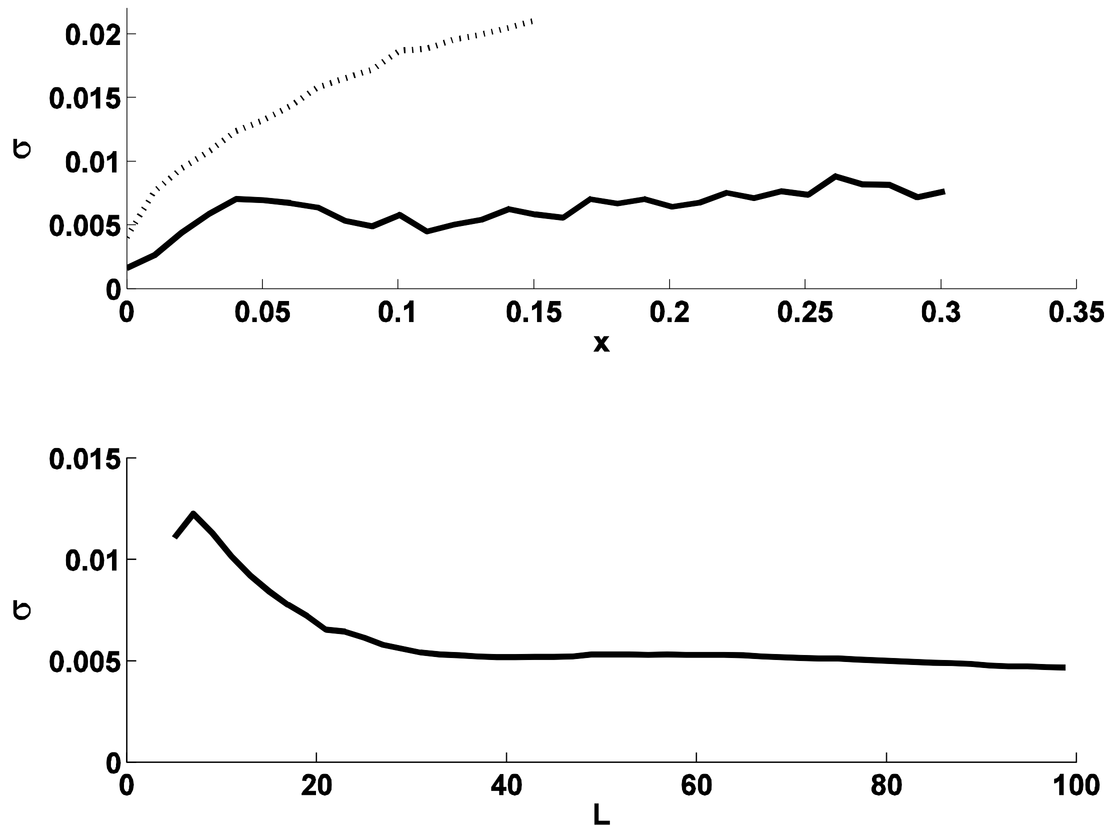
- **Level spacing distribution P(s) gives x, the fraction of missed levels. P(k;s) is k^{th} nearest level spacing**
- **Agvaanluvsan et al.Nucl. Instr. Meth. Phys. Res. A 498(2003)459-469**

$$P(s) = \sum_{k=0}^{\infty} (1-x)x^k P(k;s)$$

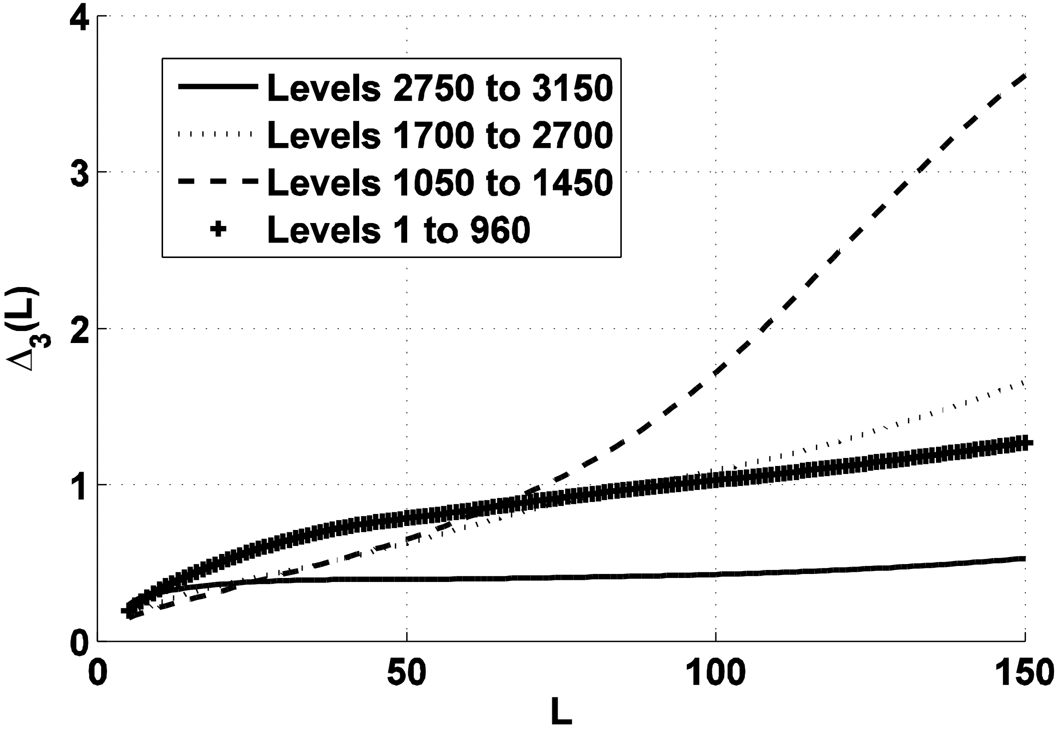
$$L = \prod_i P(s_i)$$

Find the "x" that minimizes ln L

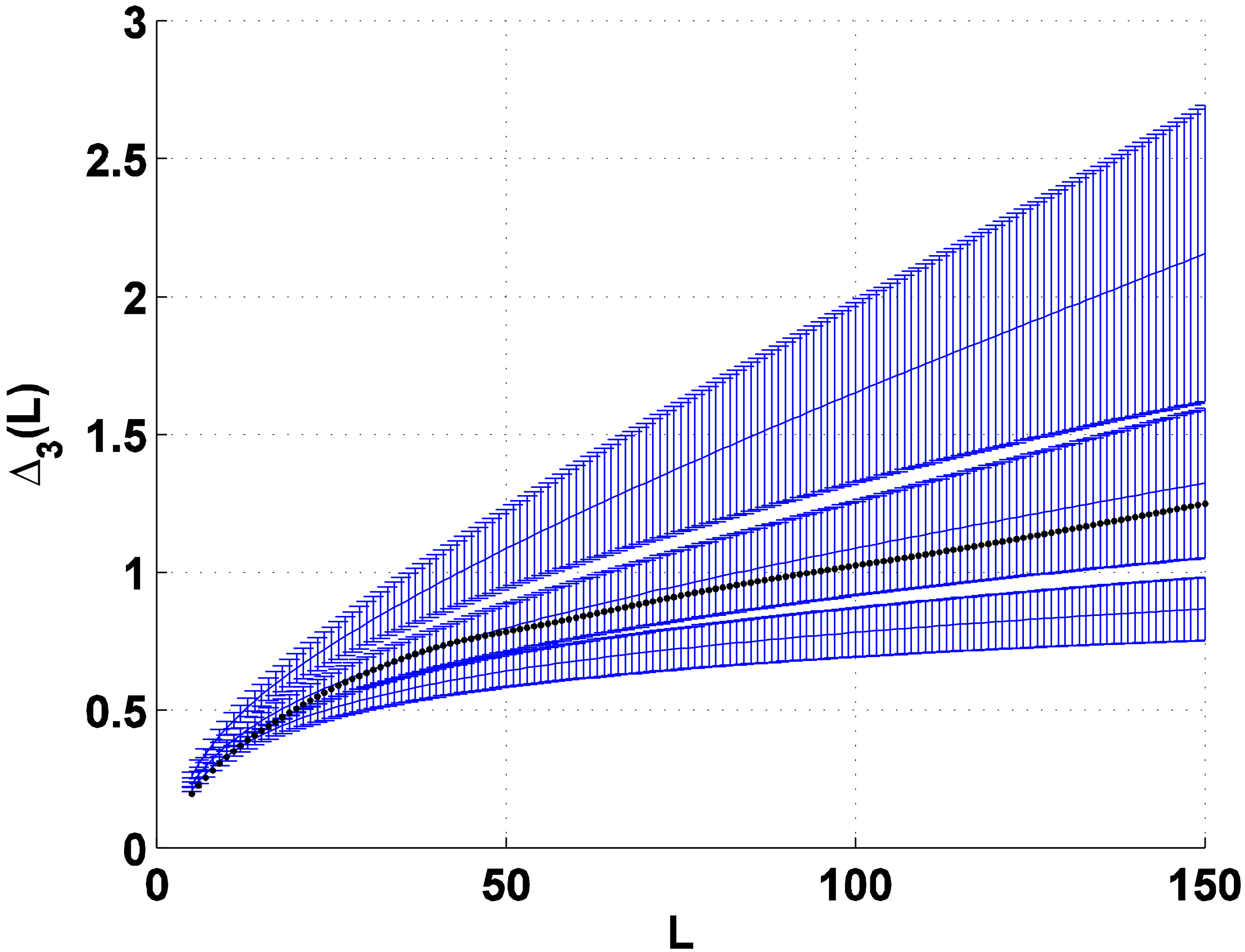
Testing the $\Delta_3(L)$ method



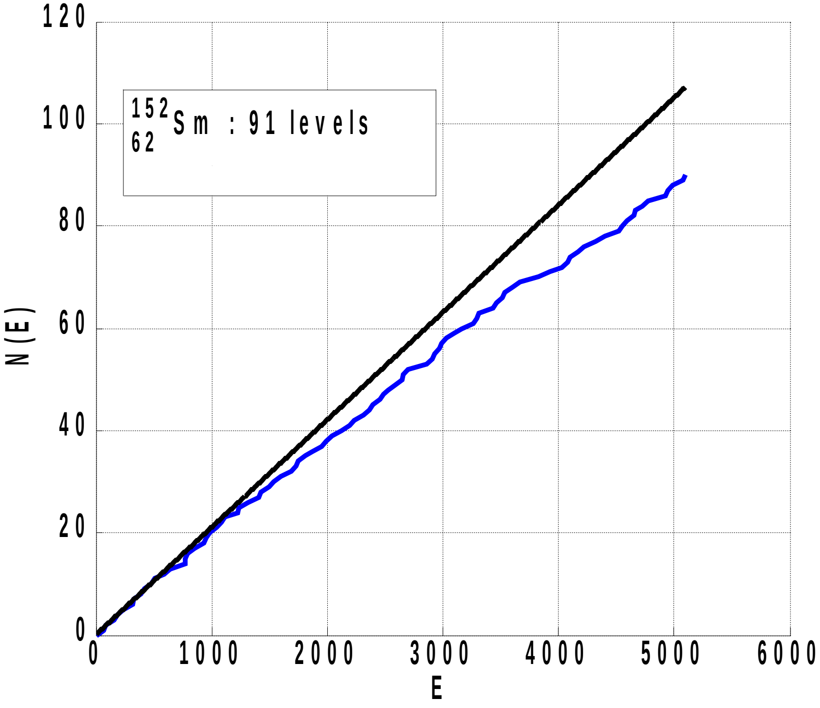
Uranium Data

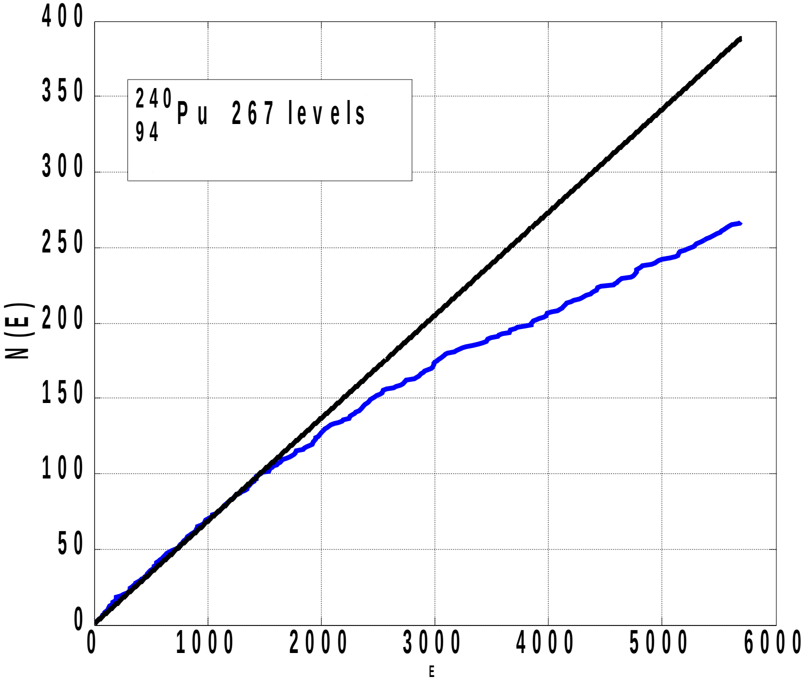


Uranium Data
lowest 960 levels 4% missed

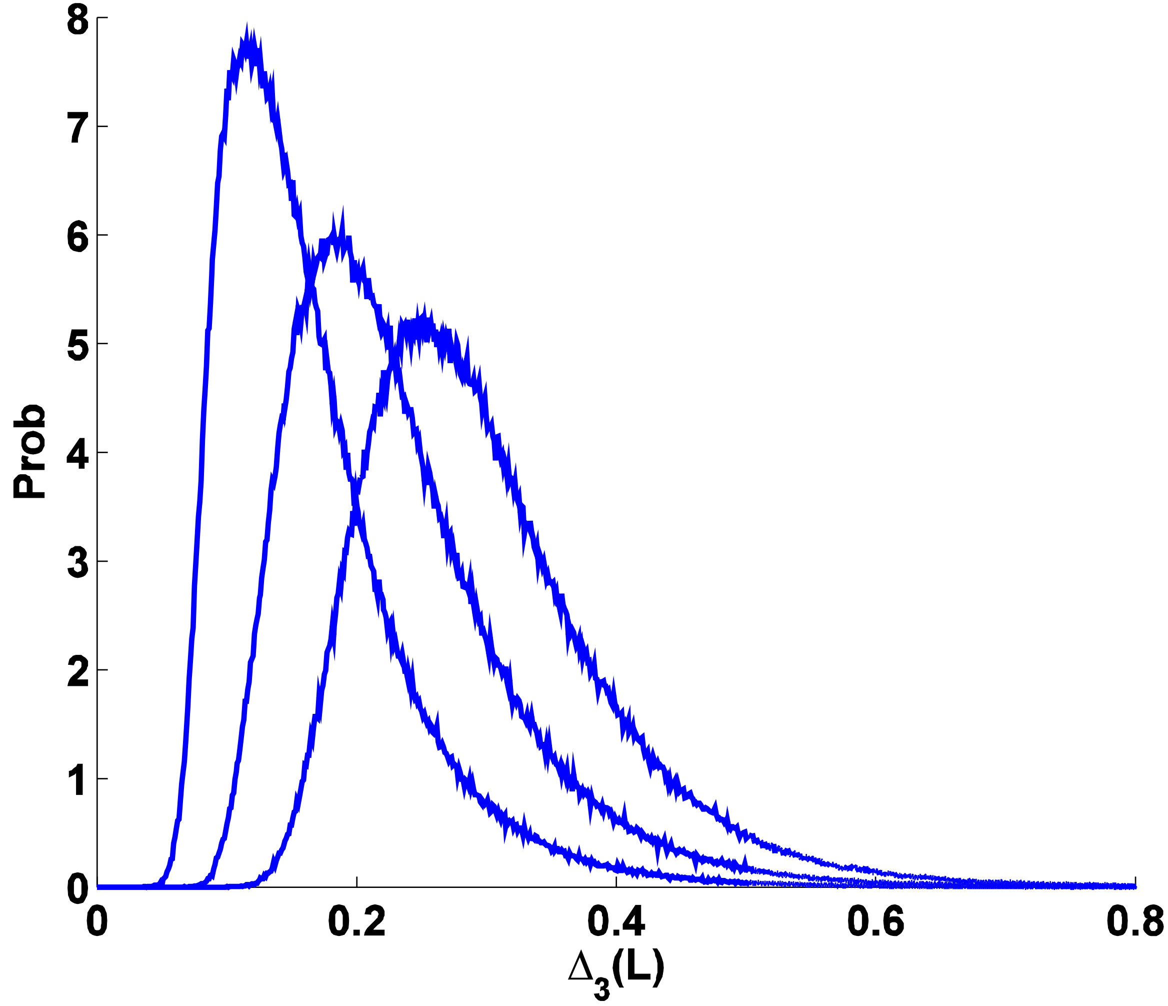


Cumulative Level Number: Raw Data

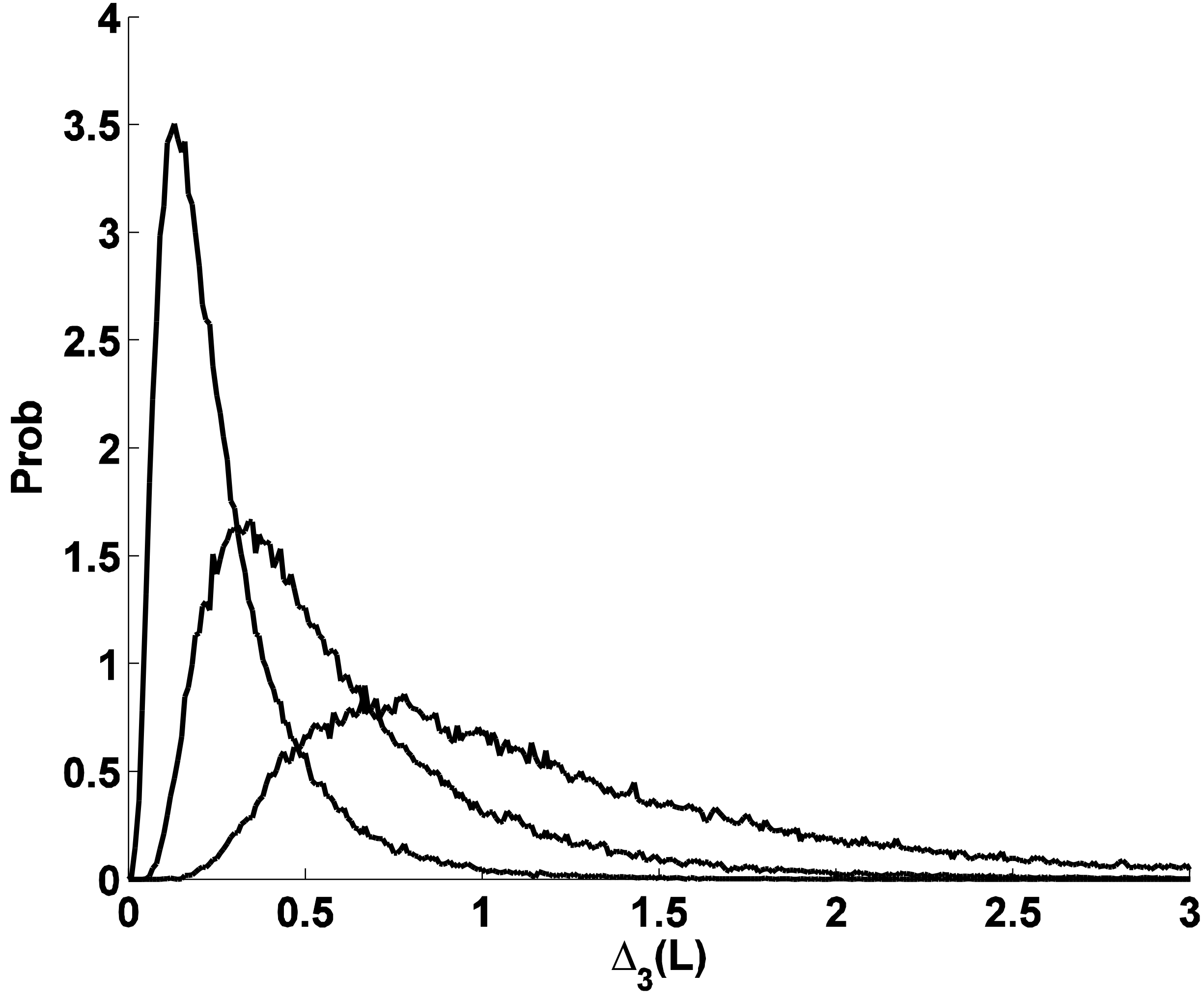




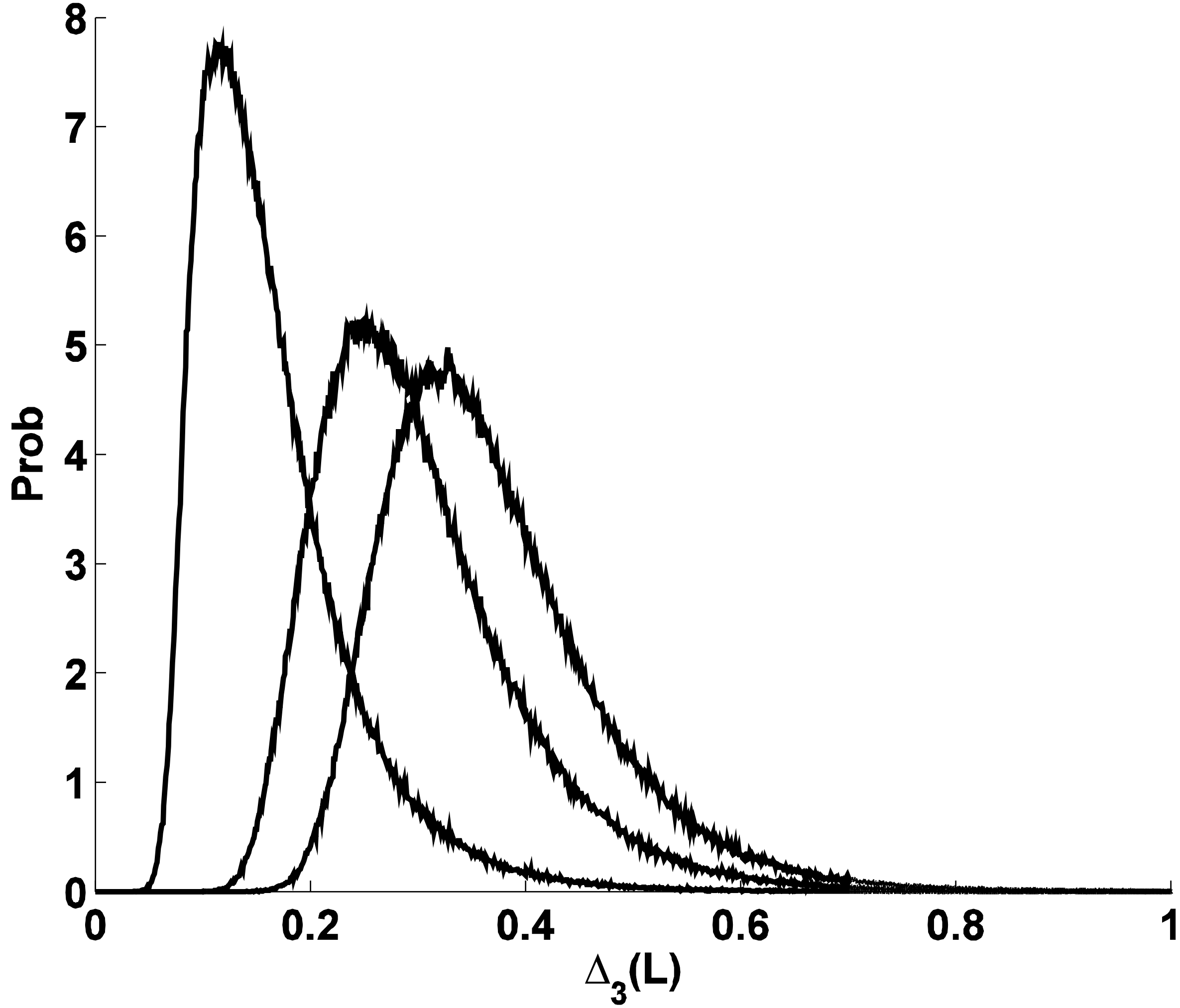
GOE L=5, 10, 20 f=1



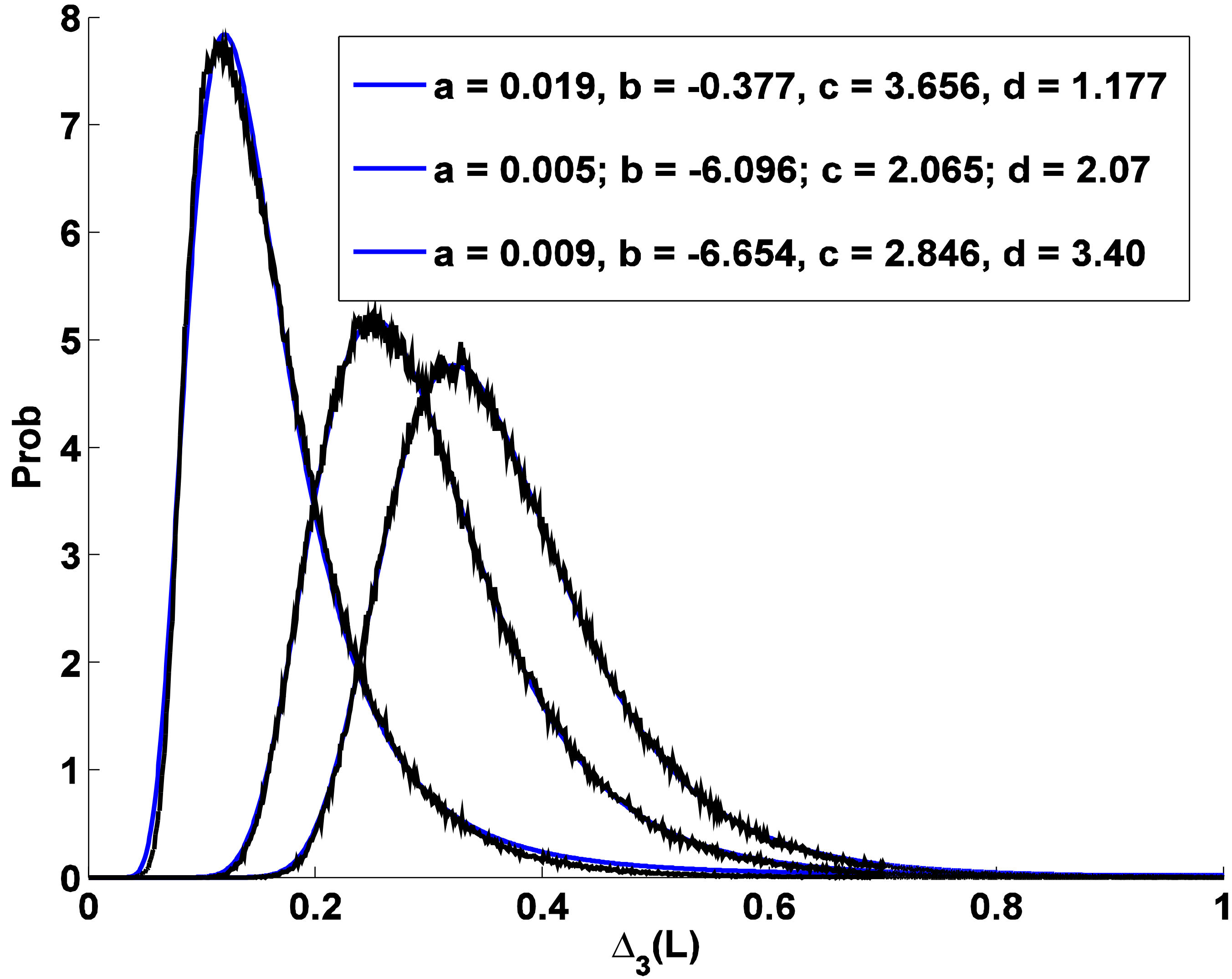
Poisson L=5, 10, 20 f=1



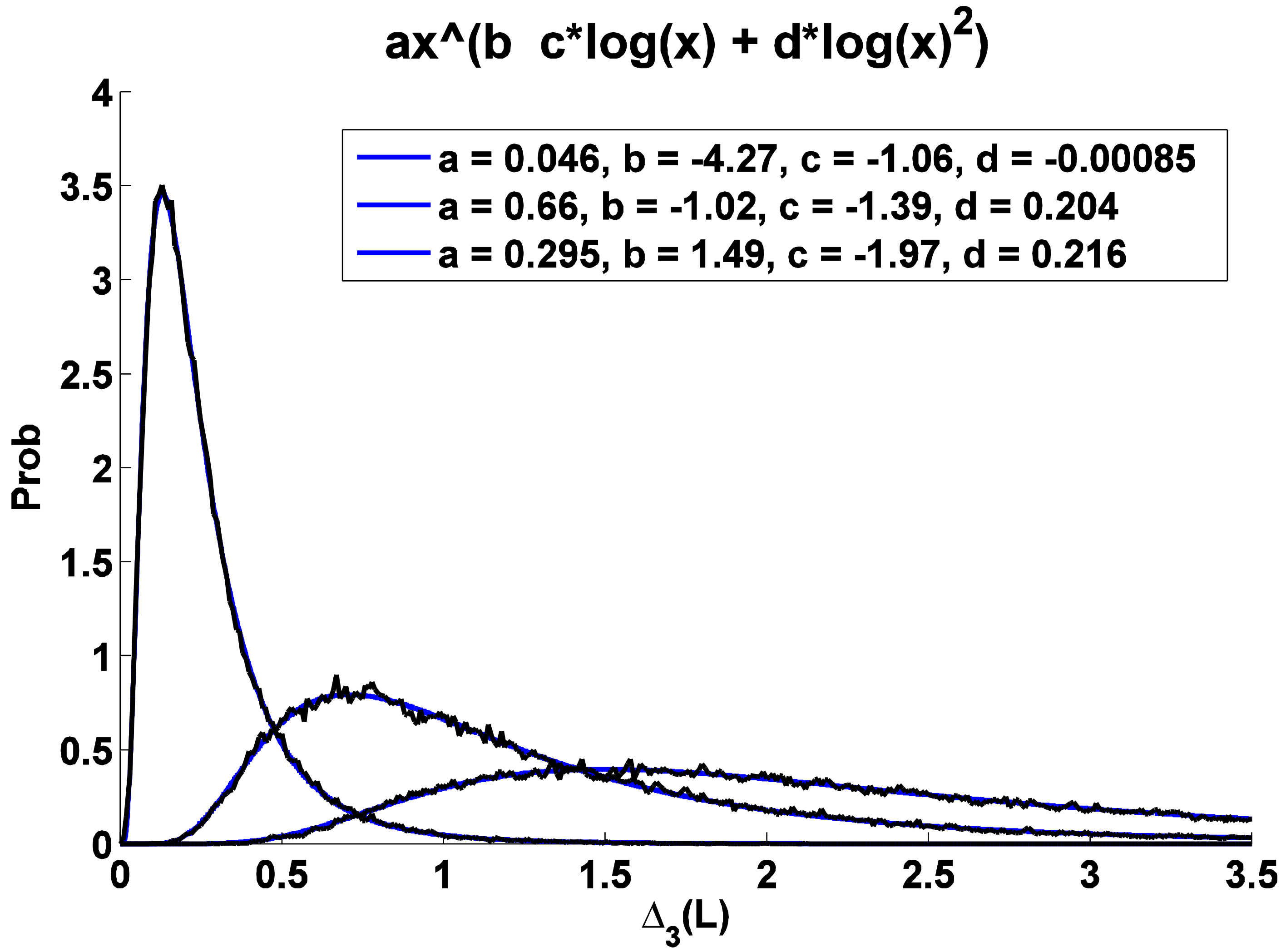
L=20; f=0, 0.05, 0.10 GOE



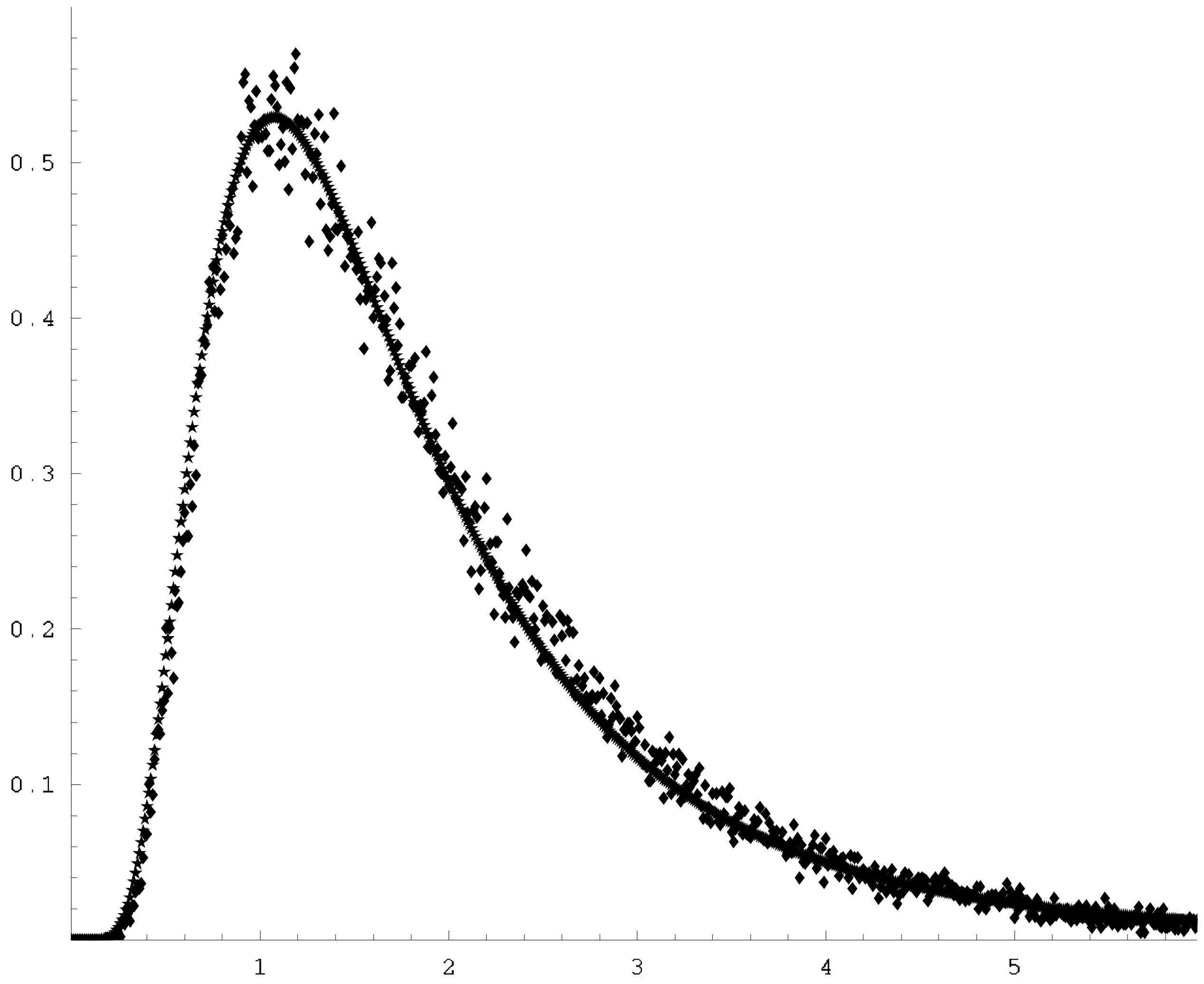
GOE L=5, 20, 40 f=1



Pois L=5, 20, 40


ax^(b c*log(x) + d*log(x)^2)
a = 0.046, b = -4.27, c = -1.06, d = -0.00085
a = 0.66, b = -1.02, c = -1.39, d = 0.204
a = 0.295, b = 1.49, c = -1.97, d = 0.216
Prob
Δ3(L)
0
0.5
1
1.5
2
2.5
3
3.5
4

Pois L=30 linear parameters



Secular Variation

- **Square well in 3-D**